ALGEBRAIC RELATIONS FOR TRIG FUNCTIONS OF THE SAME ANGLE

- Denoting the length of the hypoteneuse of a right angle triangle by h, the length of the side adjacent to the angle θ by a and the length of the side opposite to the angle θ by o, the basic geometric definitions of the six trig functions (discussed in detail in Chapter 2 of this course) become

$$\begin{aligned} sin(\theta) &= o/h, \qquad & cos(\theta) = a/h \\ tan(\theta) &= o/a, \qquad & cot(\theta) = a/o \\ sec(\theta) &= h/a, \qquad & csc(\theta) = h/o\ . \end{aligned}$$

- From these relations it is easy to see that many relations exist among the different trig functions of the same angle. Some examples, together with illustrations showing where they come from, are given below:

$$\begin{aligned} tan(\theta) &= \frac{o}{a} = \frac{o/h}{a/h} = sin(\theta)/cos(\theta) \\ &= \frac{o}{a} = \frac{1}{a/o} = 1/cot(\theta) \\ cot(\theta) &= \frac{a}{o} = \frac{a/h}{o/h} = cos(\theta)/sin(\theta) \\ &= \frac{a}{o} = \frac{1}{o/a} = 1/tan(\theta) \\ csc(\theta) &= \frac{h}{o} = \frac{1}{o/h} = 1/sin(\theta) \\ &= \frac{h}{o} = \frac{h/a}{o/a} = sec(\theta)/tan(\theta) \\ &= \left(\frac{h}{a}\right)\left(\frac{a}{o}\right) = sec(\theta)\,cot(\theta) \end{aligned}$$

$$\begin{aligned} sec(\theta) &= \frac{h}{a} = \frac{1}{a/h} = 1/cos(\theta) \\ &= \frac{h/o}{a/o} = csc(\theta)/cot(\theta) \\ &= \left(\frac{h}{o}\right)\left(\frac{o}{a}\right) = csc(\theta)\, tan(\theta) \end{aligned} \quad (1)$$

- Later in this Chapter we'll see that it's particularly easy to visualize the behavior of the sine and cosine functions geometrically, and, as a result, algebraic relations expressing the other four trig functions in terms of the sine and cosine turn out to be especally useful. These relations are

$$\begin{aligned} tan(\theta) &= sin(\theta)/cos(\theta) \\ csc(\theta) &= 1/sin(\theta) \\ sec(\theta) &= 1/cos(\theta) \\ cot(\theta) &= cos(\theta)/sin(\theta)\ . \end{aligned} \quad (2)$$

(All four of these relations were derived explicitly in Equations 1 above.)

- In the algebraic relations of Equations 2, we can think of the sine and cosine functions as the "basic" ones and the other four functions as "auxillary".
- One could in fact have chosen any two of the six functions as "basic" and written the remaining four as ratios involving them, but the choice of the sine and cosine as "basic" turns out to be the most natural one once one goes beyond the first quadrant ($0 < \theta < \pi/2$) values for θ which are possible in acutal right-angle triangles and tries to define the trig functions for other angles as well (for more on this point, see the relevant later section in this Chapter of the course). The reason is that $sin(\theta)$ and $cos(\theta)$ are not only the easiest to visualize geometrically, but are also the only two of the six functions which are defined for all possible values of the angle θ (see later in the chapter for more one this point).

EXERCISES:

Determine which of the following equations represent true "identities" (results which are true for all angles θ for which both sides of the claimed equality are defined) and which are false.

1. $sin(\theta) = 1/sec(\theta)$
2. $cos(\theta) = 1/sec(\theta)$
3. $sec(\theta)/csc(\theta) = tan(\theta)$
4. $sec(\theta)/tan(\theta) = csc(\theta)$
5. $cot(\theta)/csc(\theta) = sin(\theta)$
6. $tan(\theta)\, sec(\theta)\, [cos(\theta)]^2 = 1$
7. $sin(\theta)\, cos(\theta) \left[\frac{sec(\theta)+csc(\theta)}{sin(\theta)+cos(\theta)}\right] = 1$