ANSWERS TO EXERCISES, CHAPTER 5, SECTIONS (b), (c), (d), (e) and (f)

SECTION (b):

1. Quadrants I and II, with $cos(\theta) = 4/5$ in Quadrant I and $cos(\theta) = -4/5$ in Quadrant II.

2. Quadrants II and III, with $sin(\theta) = (\sqrt{3})/2$ in Quadrant II and $sin(\theta) = -(\sqrt{3})/2$ in Quadrant III.

3. Quadrants II and IV, with $csc(\theta) = 3$ in Quadrant II and $csc(\theta) = -3$ in Quadrant IV.

4. Quadrants III and IV, with $cot(\theta) = 3$ in Quadrant III and $cot(\theta) = -3$ in Quadrant IV.

5. Quadrants II and III, with $tan(\theta) = -5$ in Quadrant II and $tan(\theta) = 5$ in Quadrant III.

6. Quadrants I and III, with $sec(\theta) = \sqrt{25/16} = 5/4$ in Quadrant I and $sec(\theta) = -5/4$ in Quadrant III.

7. Quadrants II and III, with $cot(\theta) = -4/3$ in Quadrant II and $cot(\theta) = 4/3$ in Quadrant III.

8. Quadrants I and IV, with $cot(\theta) = 5/12$ in Quadrant I and $cot(\theta) = -5/12$ in Quadrant IV. (The tangent is $\pm\sqrt{sec^2(\theta) - 1} = \pm\sqrt{(169/25) - 1} = \pm\sqrt{144/25} = \pm 12/5$, and the cotangent is $1/tangent$.)

9. Quadrants III and IV. In Quadrant III, $cos(\theta) = -12/13$, $csc(\theta) = -13/5$, $sec(\theta) = -13/12$, $tan(\theta) = 5/12$ and $cot(\theta) = 12/5$. In Quadrant IV, $cos(\theta) = 12/13$, $csc(\theta) = -13/5$, $sec(\theta) = 13/12$, $tan(\theta) = -5/12$ and $cot(\theta) = -12/5$.

10. Quadrants II and III. In Quadrant II, $cos(\theta) = -1/\sqrt{17}$, $sin(\theta) = \sqrt{16/17}$, $csc(\theta) = \sqrt{17/16}$, $tan(\theta) = -4$ and $cot(\theta) = -1/4$. In Quadrant III, $cos(\theta) = -1/\sqrt{17}$, $sin(\theta) = -\sqrt{16/17}$, $csc(\theta) = -\sqrt{17/16}$, $tan(\theta) = 4$ and $cot(\theta) = 1/4$.

11. Quadrants I and III. In Quadrant I, $csc(\theta) = 4$, $sin(\theta) = 1/4$, $cos(\theta) = \sqrt{15/16}$, $sec(\theta) = \sqrt{16/15}$, and $tan(\theta) = 1/\sqrt{15}$. In Quadrant III, $csc(\theta) = -4$, $sin(\theta) = -1/4$, $cos(\theta) = -\sqrt{15/16}$, $sec(\theta) = -\sqrt{16/15}$, and $tan(\theta) = 1/\sqrt{15}$.

SECTION (c):

1. The trig functions of $5\pi/12$ are

$$sin\left(\frac{5\pi}{12}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}, \qquad cos\left(\frac{5\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$csc\left(\frac{5\pi}{12}\right) = \frac{2\sqrt{2}}{1+\sqrt{3}}, \qquad sec\left(\frac{5\pi}{12}\right) = \frac{2\sqrt{2}}{\sqrt{3}-1}$$

$$tan\left(\frac{5\pi}{12}\right) = \frac{1+\sqrt{3}}{\sqrt{3}-1}, \qquad cot\left(\frac{5\pi}{12}\right) = \frac{\sqrt{3}-1}{1+\sqrt{3}} \ .$$

2. Since $\pi/5 = \frac{1}{2}(2\pi/5)$,

$$cos\left(\frac{\pi}{5}\right) = \sqrt{\frac{1+\left[(\sqrt{5}-1)/4\right]}{2}} = \sqrt{\frac{3+\sqrt{5}}{8}}$$

$$sin\left(\frac{\pi}{5}\right) = \sqrt{\frac{1-\left[(\sqrt{5}-1)/4\right]}{2}} = \sqrt{\frac{5-\sqrt{5}}{8}} \ .$$

3. Using the general identities $sin(\pi-\theta) = sin(\theta)$ and $cos(\pi-\theta) = -cos(\theta)$, with $\theta = \pi/12$,

$$sin\left(\frac{11\pi}{12}\right) = sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}, \qquad cos\left(\frac{11\pi}{12}\right) = -cos\left(\frac{11\pi}{12}\right) = -\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) \ .$$

4. With $3\pi/8 = (\pi/2) - (\pi/8)$, the general identities $sin\left(\frac{\pi}{2} - \theta\right) = cos(\theta)$, $cos\left(\frac{\pi}{2} - \theta\right) = sin(\theta)$, with $\theta = \pi/8$, yield

$$sin\left(\frac{3\pi}{8}\right) = cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$
$$cos\left(\frac{3\pi}{8}\right) = sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \; .$$

(An alternate approach is to note that $3\pi/8 = (\pi/4) + (\pi/8)$, and use the addition formulas:

$$sin(3\pi/8) = sin(\pi/4)cos(\pi/8) + cos(\pi/4)sin(\pi/8) = \left(\frac{1}{\sqrt{2}}\right)\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} + \left(\frac{1}{\sqrt{2}}\right)\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$
$$cos(3\pi/8) = cos(\pi/4)cos(\pi/8) - sin(\pi/4)sin(\pi/8) = \left(\frac{1}{\sqrt{2}}\right)\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} - \left(\frac{1}{\sqrt{2}}\right)\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \; .$$

Although the two sets of answers look different, it is possible to show that they are in fact the same. The easiest way to do this is to square each of the results just obtained and verify that they match the squares of the simpler expressions obtained earlier.)

SECTION (d):

1. $sec(\pi+\theta) = -sec(\theta)$, $csc(\pi+\theta) = -csc(\theta)$ and $cot(\pi+\theta) = cot(\theta)$.
2. $sec(-\theta) = sec(\theta)$, $csc(-\theta) = -csc(\theta)$, $tan(-\theta) = -tan(\theta)$ and $cot(-\theta) = -cot(\theta)$.

3. $7\pi/5 = \pi + (2\pi/5)$. The general identities $sin(\pi+\theta) = -sin(\theta)$ and $cos(\pi+\theta) = -cos(\theta)$, specialized to $\theta = 2\pi/5$, thus imply:

$$sin\left(\frac{7\pi}{5}\right) = -\,sin\left(\frac{2\pi}{5}\right) = -\sqrt{\frac{5+\sqrt{5}}{8}}$$
$$cos\left(\frac{7\pi}{5}\right) = -\,cos\left(\frac{2\pi}{5}\right) = -\left(\frac{\sqrt{5}-1}{4}\right) \;,$$

which in turn imply

$$sec\left(\frac{7\pi}{5}\right) = \frac{1}{cos\,(7\pi/5)} = \frac{-4}{\sqrt{5}-1}$$
$$tan\left(\frac{7\pi}{5}\right) = \frac{sin\,(7\pi/5)}{cos\,(7\pi/5)} = \sqrt{\frac{5+\sqrt{5}}{8}}\left(\frac{4}{\sqrt{5}-1}\right) \;.$$

4. $8\pi/5 = 2\pi - (2\pi/5)$, which is physically equivalent to $-2\pi/5$. Thus

$$\begin{aligned}
sin\left(\frac{8\pi}{5}\right) &= sin\left(-\frac{2\pi}{5}\right) = -\,sin\left(\frac{2\pi}{5}\right) = -\sqrt{\frac{5+\sqrt{5}}{8}} \\
cos\left(\frac{8\pi}{5}\right) &= cos\left(-\frac{2\pi}{5}\right) = cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4} \\
csc\left(\frac{8\pi}{5}\right) &= \frac{1}{sin\,(8\pi/5)} = -\sqrt{\frac{8}{5+\sqrt{5}}} \\
cot\left(\frac{8\pi}{5}\right) &= \frac{cos\,(8\pi/5)}{sin\,(8\pi/5)} = -\left(\frac{\sqrt{5}-1}{4}\right)\sqrt{\frac{8}{5+\sqrt{5}}}\ .
\end{aligned}$$

5. $5\pi/3 = 2\pi - (\pi/3)$, which is physically equivalent to $-\pi/3$. Thus

$$\begin{aligned}
sin\left(\frac{5\pi}{3}\right) &= sin\left(-\frac{\pi}{3}\right) = -\,sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \\
cos\left(\frac{5\pi}{3}\right) &= cos\left(-\frac{\pi}{3}\right) = cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \\
csc\left(\frac{5\pi}{3}\right) &= \frac{1}{sin\,(5\pi/3)} = \frac{-2}{\sqrt{3}} \\
cot\left(\frac{5\pi}{3}\right) &= \frac{cos\,(5\pi/3)}{sin\,(5\pi/3)} = \frac{-1}{\sqrt{3}}\ .
\end{aligned}$$

6. $5\pi/4 = \pi + (\pi/4)$, which implies

$$\begin{aligned} sin\left(\frac{5\pi}{4}\right) &= -\,sin\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}} \\ cos\left(\frac{5\pi}{4}\right) &= -\,cos\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}} \\ sec\left(\frac{5\pi}{4}\right) &= \frac{1}{cos\,(5\pi/4)} = -\sqrt{2} \\ cot\left(\frac{5\pi}{4}\right) &= \frac{cos\,(5\pi/4)}{sin\,(5\pi/4)} = 1\ . \end{aligned}$$

7. The general identities $sin(-\theta) = -sin(\theta)$, $cos(-\theta) = cos(\theta)$, applied to $\theta = 5\pi/4$, yield

$$\begin{aligned} sin\left(-\frac{5\pi}{4}\right) &= -\,sin\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}} \\ cos\left(-\frac{5\pi}{4}\right) &= cos\left(\frac{5\pi}{4}\right) = \frac{-1}{\sqrt{2}} \\ csc\left(-\frac{5\pi}{4}\right) &= \frac{1}{sin\,(-5\pi/4)} = \sqrt{2} \\ tan\left(-\frac{5\pi}{4}\right) &= \frac{sin\,(-5\pi/4)}{cos\,(-5\pi/4)} = -1\ . \end{aligned}$$

SECTION (e):

1. The sine and cosine of $13\pi/8 = (3\pi/2) + (\pi/8)$ are

$$sin(13\pi/8) = sin(3\pi/2)cos(\pi/8) + cos(3\pi/2)sin(\pi/8) = -cos(\pi/8) = -\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$cos(13\pi/8) = cos(3\pi/2)cos(\pi/8) - sin(3\pi/2)sin(\pi/8) = sin(\pi/8) = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \ .$$

2. The sine and cosine of $9\pi/10 = (\pi/2) + (2\pi/5)$ are

$$sin(9\pi/10) = sin(\pi/2)cos(2\pi/5) + cos(\pi/2)sin(2\pi/5) = cos(2\pi/5) = \frac{\sqrt{5}-1}{4}$$

$$cos(9\pi/10) = cos(\pi/2)cos(2\pi/5) - sin(\pi/2)sin(2\pi/5) = -sin(2\pi/5) = -\sqrt{\frac{5+\sqrt{5}}{8}} \ .$$

3. With $11\pi/10 = (3\pi/2) - (2\pi/5)$,

$$sin(11\pi/10) = sin(3\pi/2)cos(2\pi/5) - cos(3\pi/2)sin(2\pi/5) = -cos(2\pi/5) = -\left(\frac{\sqrt{5}-1}{4}\right)$$

$$cos(11\pi/10) = cos(3\pi/2)cos(2\pi/5) + sin(3\pi/2)sin(2\pi/5) = -sin(2\pi/5) = -\sqrt{\frac{5+\sqrt{5}}{8}}$$

$$sec(11\pi/10) = \frac{1}{cos(11\pi/10)} = -\sqrt{\frac{8}{5+\sqrt{5}}}$$

$$tan(11\pi/10) = \frac{sin(11\pi/10)}{cos(11\pi/10)} = \left(\frac{\sqrt{5}-1}{4}\right)\sqrt{\frac{8}{5+\sqrt{5}}} \ .$$

4. The sine and cosine of $\pi/60 = (\pi/10) - (\pi/12)$ are

$$\begin{aligned} sin\left(\frac{\pi}{60}\right) &= sin\left(\frac{\pi}{10}\right) cos\left(\frac{\pi}{12}\right) - cos\left(\frac{\pi}{10}\right) sin\left(\frac{\pi}{12}\right) \\ &= \left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) - \sqrt{\frac{5+\sqrt{5}}{8}}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) \\ cos\left(\frac{\pi}{60}\right) &= cos\left(\frac{\pi}{10}\right) cos\left(\frac{\pi}{12}\right) + sin\left(\frac{\pi}{10}\right) sin\left(\frac{\pi}{12}\right) \\ &= \sqrt{\frac{5+\sqrt{5}}{8}}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) + \left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) . \end{aligned}$$

5. Since $19\pi/12 = (3\pi/2) + (\pi/12)$

$$sin\left(\frac{19\pi}{12}\right) = sin\left(\frac{3\pi}{2}\right) cos\left(\frac{\pi}{12}\right) + cos\left(\frac{3\pi}{2}\right) sin\left(\frac{\pi}{12}\right) = -cos\left(\frac{\pi}{12}\right) = -\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

$$cos\left(\frac{19\pi}{12}\right) = cos\left(\frac{3\pi}{2}\right) cos\left(\frac{\pi}{12}\right) - sin\left(\frac{3\pi}{2}\right) sin\left(\frac{\pi}{12}\right) = sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}} .$$

6. With θ a first quadrant angle, $cos(\theta) = +\sqrt{1 - sin^2(\theta)} = 4/5$ and thus

$$\begin{aligned} sin(\theta + (\pi/4)) &= sin(\theta)cos(\pi/4) + cos(\theta)sin(\pi/4) = 7/(5\sqrt{2}) \\ cos(\theta + (\pi/4)) &= cos(\theta)cos(\pi/4) - sin(\theta)sin(\pi/4) = 1/(5\sqrt{2}) \\ csc(\theta + (\pi/4)) &= 1/sin(\theta + (\pi/4)) = 5\sqrt{2}/7 \\ cot(\theta + (\pi/4)) &= cos(\theta + (\pi/4))/sin(\theta + (\pi/4)) = 1/7 \ . \end{aligned}$$

7. Letting $\theta_1 = \theta_2 = \theta$,

$$\begin{aligned} sin(\theta_1 + \theta_2) = sin(\theta_1)cos(\theta_2) + cos(\theta_1)sin(\theta_2) &\Rightarrow sin(\theta + \theta) = sin(\theta)cos(\theta) + cos(\theta)sin(\theta) \\ &\Rightarrow sin(2\theta) = 2sin(\theta)cos(\theta) \\ cos(\theta_1 + \theta_2) = cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2) &\Rightarrow cos(\theta + \theta) = cos(\theta)cos(\theta) - cos(\theta)sin(\theta) \\ &\Rightarrow cos(2\theta) = cos^2(\theta) - sin^2(\theta) \ , \end{aligned}$$

as claimed.

8. If we write $3\theta = 2\theta + \theta$, using the addition formulas implies

$$\begin{aligned} sin(3\theta) &= sin(2\theta)cos(\theta) + cos(2\theta)sin(\theta) \\ cos(3\theta) &= cos(2\theta)cos(\theta) - sin(2\theta)sin(\theta) \ . \end{aligned}$$

Using the results of Exercise 7 for $sin(2\theta)$ and $cos(2\theta)$, we find

$$\begin{aligned} sin(3\theta) &= [2sin(\theta)cos(\theta)]\ cos(\theta) + \left[cos^2(\theta) - sin^2(\theta)\right]\ sin(\theta) = 3cos^2(\theta)sin(\theta) - sin^3(\theta) \\ cos(3\theta) &= \left[cos^2(\theta) - sin^2(\theta)\right]\ cos(\theta) - [2sin(\theta)cos(\theta)]\ sin(\theta) = cos^3(\theta) - 3cos(\theta)sin^2(\theta) \ . \end{aligned}$$

SECTION (f):

1. Using the tangent addition formulas

$$tan(\phi + \phi') = \frac{3 + (-1)}{1 - (3)(-1)} = \frac{2}{4} = \frac{1}{2}$$
$$tan(\phi - \phi') = \frac{3 - (-1)}{1 + (3)(-1)} = \frac{4}{(-2)} = -2 \ .$$

2. The answers are the same as in Exercise 1. The tangent addition formulas are valid regardless of what quadrant the angles lie in. As soon as one has the values of $tan(\phi)$ and $tan(\phi')$, this is enough to determine $tan(\phi \pm \phi')$ for all possible quadrant combinations.

3. Using the tangent addition formulas, plus $cot(\theta) = 1/tan(\theta)$,

$$tan(\phi + \phi') = \frac{(1/2) + 1}{1 - (1/2)(1)} = \frac{3/2}{1/2} = 3 \ \Rightarrow \ cot(\phi + \phi') = 1/3$$
$$tan(\phi - \phi') = \frac{(1/2) - 1}{1 + (1/2)(1)} = \frac{(-1/2)}{3/2} = -1/3 \ \Rightarrow \ cot(\phi + \phi') = -3 \ .$$

4. Using the double-angle formulas,

$$sin(2\phi) = 2sin(\phi)cos(\phi) = -120/169$$
$$cos(2\phi) = cos^2(\phi) - sin^2(\phi) = -119/169$$
$$csc(2\phi) = 1/sin(2\phi) = -169/120$$
$$sec(2\phi) = 1/cos(2\phi) = -169/119$$
$$tan(2\phi) = sin(2\phi)/cos(2\phi) = 120/119$$
$$cot(2\phi) = cos(2\phi)/sin(2\phi) = 119/120 \ .$$

5. Using the half-angle formulas, and remembering that, for a second quadrant angle ϕ, $\phi/2$ will lie in the first quadrant, and hence have positive values for both its sine and cosine,

$$
\begin{aligned}
sin(\phi/2) &= \sqrt{\frac{1-cos(\phi)}{2}} = \sqrt{\frac{9}{13}} = \frac{3}{\sqrt{13}} \\
cos(\phi/2) &= \sqrt{\frac{1+cos(\phi)}{2}} = \sqrt{\frac{4}{13}} = \frac{2}{\sqrt{13}} \\
csc(\phi/2) &= 1/sin(\phi/2) = \sqrt{\frac{13}{9}} = \frac{\sqrt{13}}{3} \\
sec(\phi/2) &= 1/cos(\phi/2) = \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2} \\
tan(\phi/2) &= sin(\phi/2)/cos(\phi/2) = \sqrt{\frac{9}{4}} = \frac{3}{2} \\
cot(\phi/2) &= cos(\phi/2)/sin(\phi/2) = \sqrt{\frac{4}{9}} = \frac{2}{3}\ .
\end{aligned}
$$

6. Since $\pi/24 = (1/2)(\pi/12)$, we can use the half-angle formulas, which yield

$$
\begin{aligned}
sin\left(\frac{\pi}{24}\right) &= \sqrt{\frac{1-cos(\pi/12)}{2}} = \sqrt{\frac{1-\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{2}} = \sqrt{\frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}} \\
cos\left(\frac{\pi}{24}\right) &= \sqrt{\frac{1+cos(\pi/12)}{2}} = \sqrt{\frac{1+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{2}} = \sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}}\ .
\end{aligned}
$$

The cotangent and secant values follows from these:

$$
sec\left(\frac{\pi}{24}\right) = \frac{1}{cos\,(\pi/24)} = \sqrt{\frac{4\sqrt{2}}{2\sqrt{2}+\sqrt{3}+1}}
$$

$$cot\left(\frac{\pi}{24}\right) = \frac{cos\,(\pi/24)}{sin\,(\pi/24)} = \sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{2\sqrt{2}-\sqrt{3}-1}}\ .$$

(Another solution would be to note that $\pi/24 = (\pi/8) - (\pi/12)$ and use the previously worked out values of the sines and cosines of $\pi/8$ and $\pi/12$ as input to the addition formulas to get $sin(\pi/24)$ and $cos(\pi/24)$.)

7. Since $5\pi/8 = (1/2)(5\pi/4)$, we can again use the half-angle formulas, which yield, taking into account that $5\pi/8$ is a second quadrant angle, and hence has a positive sine, but negative cosine,

$$sin\left(\frac{5\pi}{8}\right) = \sqrt{\frac{1-cos(5\pi/4)}{2}} = \sqrt{\frac{1-(-1/\sqrt{2})}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$cos\left(\frac{5\pi}{8}\right) = -\sqrt{\frac{1+cos(5\pi/4)}{2}} = -\sqrt{\frac{1+(-1/\sqrt{2})}{2}} = -\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}\ .$$

The cotangent and cosecant values follow from these:

$$csc\left(\frac{5\pi}{8}\right) = \frac{1}{sin\,(5\pi/8)} = \sqrt{\frac{2\sqrt{2}}{\sqrt{2}+1}}$$

$$cot\left(\frac{5\pi}{8}\right) = \frac{cos\,(5\pi/8)}{sin\,(5\pi/8)} = -\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}\ .$$

(Another solution, which doesn't use the information about the values of the sine and cosine of $5\pi/4$, would be to note that $5\pi/8 = (\pi/2) + (\pi/8)$ and use the addition formulas, with the sine and cosine values of $\pi/2$ and $\pi/8$ as input.)

8. $sin(5t)cos(4t) = \frac{1}{2}\,[sin(9t) + sin(t)]$.

9. $sin(4t)cos(5t) = \frac{1}{2}\,[sin(9t) + sin(-t)] = \frac{1}{2}\,[sin(9t) - sin(t)]$.

10. $sin(4t)sin(5t) = \frac{1}{2}\,[cos(4t-5t) - cos(4t+5t)] = \frac{1}{2}\,[cos(t) - cos(9t)]$.

11. $cos(4t)cos(5t) = \frac{1}{2}\,[cos(4t+5t) + cos(4t-5t)] = \frac{1}{2}\,[cos(9t) + cos(t)]$.