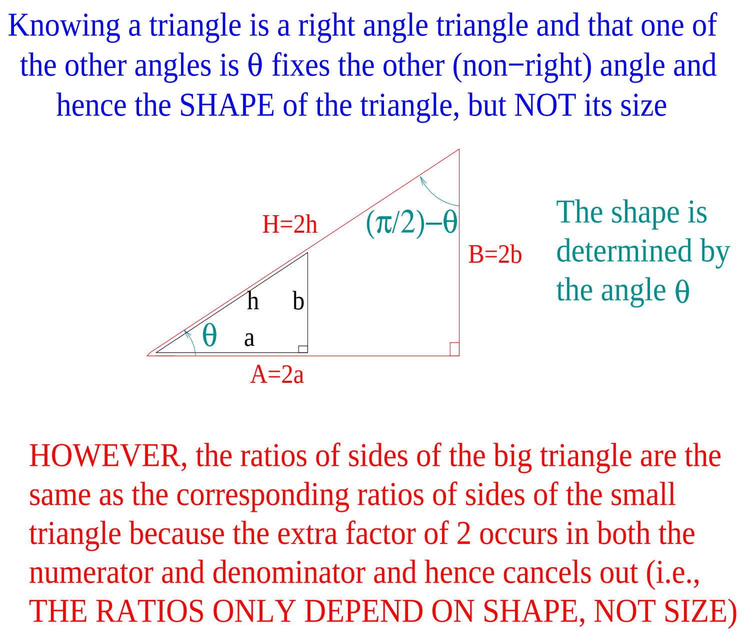
THE BASIC GEOMETRICAL MEANING OF THE SIX TRIG FUNCTIONS

Recall that, in a right angle triangle, giving one of the two non-right angles in the triangle fixes the second such angle. Thus, the SHAPE (set of angles) of a right angle triangle is entirely fixed by giving the value of just one of the two non-right angles in the triangle. An example of two triangles with the same shape, but different sizes (the larger one having sides twice as large as the corresponding sides of the smaller one), is shown in Figure 1.

FIG. 1:



THE GEOMETRY OF THE TRIG FUNCTIONS (1^{st} QUADRANT ANGLES)

- There are six basic trigonometric (trig) functions, the sine, the cosine, the tangent, the cosecant, the secant and the cotangent.

- These functions are conventionally written, in somewhat shortened form, as $sin(\theta)$, $cos(\theta)$, $tan(\theta)$, $csc(\theta)$, $sec(\theta)$, and $cot(\theta)$.

- Each of the six functions depends on a single variable, which one should interpret physically as an angle, and which we will write as θ, ϕ, $\cdots$ below.

- The most basic meaning of each of the six trigonometric functions of the angle θ is as the appropriate ratio of two different sides of a right-angle triangle having θ as one of its non-right angles.

- Since each of the two non-right angles in a right angle triangle has a value between 0 and $\pi/2$ radians (due to the constraint that the sum of all three interior angles is π radians) this picture of the trig functions involves angles which lie in the first quadrant (we will see, in the next Chapter of the course, how to generalize the trig functions to angles which do not lie in the first quadrant).

- In the usual terminology of functions, the argument, θ, of the six trig functions (the angle on which each of the trig functions depends) is the "independent variable" and the value of the function in question is "dependent variable".

- The following points are basic ones for understanding the geometric meaning of the trig functions, and why it is that there are six of these functions.

  - The reason there are six trig functions is that there are six possible ratios of different sides of a right-angle triangle (the denominator in the ratio can be any of the three sides, which then leaves two choices for the different side to put into the numerator).

  - As noted above, once we know we have a right-angle triangle, specifying one of the two non-right angles in the triangle, θ, determines the other (which has to be $\frac{\pi}{2} - \theta$), and thus completely specifies the *shape* of the triangle since the third angle is the right angle $\pi/2$ radians.

  - While specifying one of the two non-right angles in a given right-angle triangle specifies the shape, it, of course, does not specify the size, since we can make a new triangle with the same shape but different size by rescaling all three sides by the same amount.

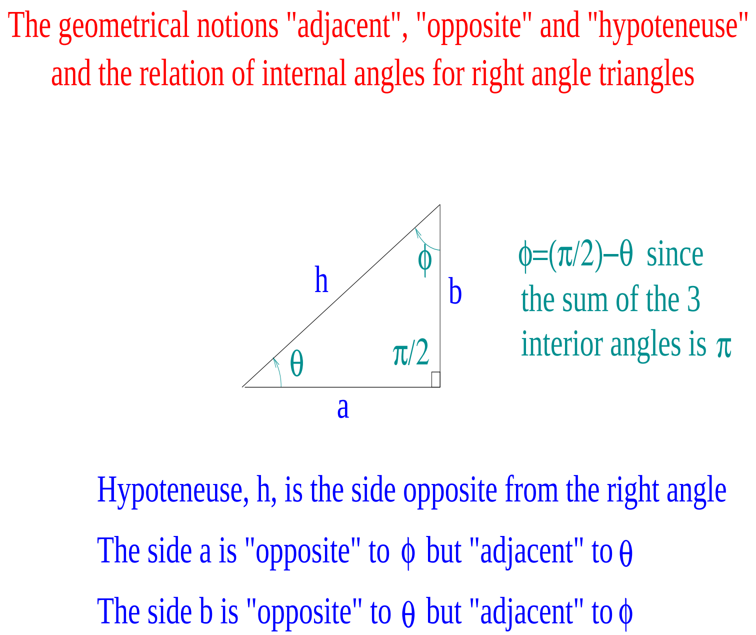
– Given two right-angle triangles with the same shape but different sizes, the lengths of the sides of the bigger version are all the same multiple (> 1) of the lengths of the corresponding sides of the smaller version. (An example, in which the lengths of the sides of the larger triangle are all a factor of 2 larger than the corresponding sides of the small triangle is shown in Fig. 1.)

– Thus, when we form a ratio of two sides of the bigger triangle, this multiple cancels out, and the ratio has the same value as the ratio of the corresponding sides of the smaller triangle.

– This argument leads to the following basic observation concerning such ratios, and hence about the trig functions:

> The ratios of the sides of a right-angle triangle, and hence the six basic trigonometric functions, depend only on the shape of the triangle, and not the particular size of the triangle having that shape. This is why the trig functions, which are just special names for the various possible ratios, depend only on one angle, the non-right angle one uses to specify the shape of the right-angle triangle in question.

FIG. 2:



– There is a conventional geometric way of characterizing what ratio of sides it is we are talking about, and hence of giving names to each of the six possible ratios of different sides of a right-angle triangle. This involves the geometrically well-defined notions of "hypoteneuse", "adjacent" and "opposite" (see Figure 2)

  * The "hypoteneuse" in a right-angle triangle is the side opposite the right angle. Since the right-angle is the biggest angle in the triangle (the other two angles have to add up to $\pi/2$ radians, or a right angle), the hypoteneuse is the longest side of the triangle.

  * There are two non-right angles in a given right-angle triangle, either of which can be used

to specify the shape of the triangle. If we chose to focus on one of these angles and call it θ, then, apart from the hypoteneuse, there is one side "adjacent" to θ and one side "opposite" to θ, as shown in Fig. 2.

* Note that the side opposite to θ is adjacent to the other non-right angle, $\phi = \frac{\pi}{2} - \theta$ and, similarly, the side adjacent to θ is opposite to $\frac{\pi}{2} - \theta$, so you have to FIRST specify which of the two non-right angles in the triangle you are going to focus on before you will know what is meant by a geometrically-characterized ratio like $adjacent/hypoteneuse$ or $opposite/adjacent$.

* Having focussed on one of the two non-right angles, θ, in the right angle triangle, we now can straightforwardly describe the three sides of the triangle as being either the hypoteneuse, the side opposite to θ, or the side adjacent to θ.

* With this geometric characterization, the 6 possible ratios are NAMED (DEFINED) as follows:

$$\begin{aligned} sin(\theta) &= opposite/hypoteneuse \\ cos(\theta) &= adjacent/hypoteneuse \\ tan(\theta) &= opposite/adjacent \\ csc(\theta) &= hypoteneuse/opposite \\ sec(\theta) &= hypoteneuse/adjacent \\ cot(\theta) &= adjacent/opposite\ . \end{aligned}$$

  These geometric definitions represent the most basic meaning of the trig functions.

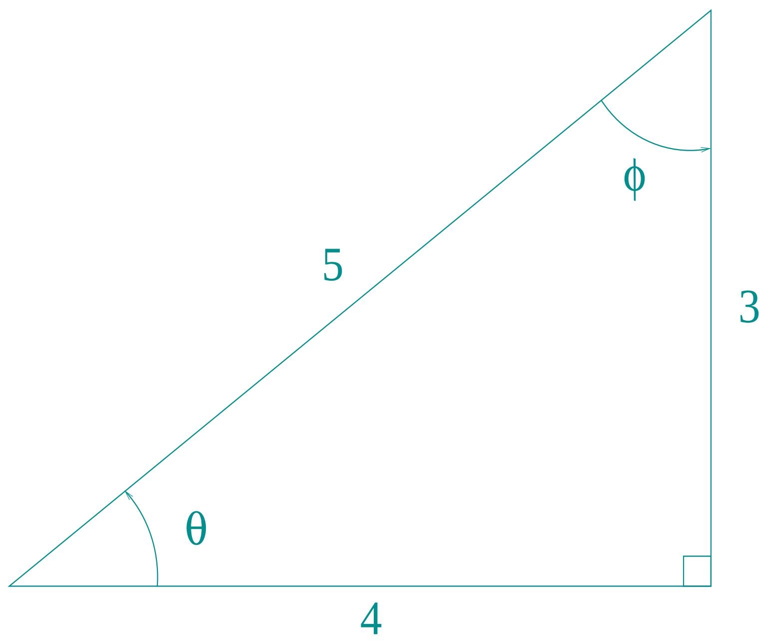
* Note that, from these definitions, if we write h for the length of the hypoteneuse, a for the length of the the adjacent side, and o for the length of the opposite side, we have also

$$\begin{aligned} a &= h\, cos(\theta) \\ o &= h\, sin(\theta) \\ o &= a\, tan(\theta),\ \ etc. \end{aligned}$$

EXERCISES

1. Compute all six trig functions of the angle θ in Figure 3.
2. Compute all six trig functions of the angle ϕ in Figure 3.

FIG. 3:



3. Compute all six trig functions of the angle α in Figure 4.
4. Compute all six trig functions of the angle β in Figure 4.

FIG. 4: