AN ALTENATE "FLOPPED TRIANGLE" PERSPECTIVE FOR THE GENERALIZATION OF THE TRIG FUNCTIONS TO NON-FIRST-QUADRANT ANGLES

The background which will be needed to follow the material in this section of the course is as follows:

- *Familiarity with the basic geometrical definitions of the various trig functions in terms of the "hypoteneuse", "adjacent" and "opposite" sides of a right-angle triangle (reviewable by reading (or re-reading) Section (c) of Chapter 2 of this course)*
- *Familiarity with the unit circle picture for the sine and cosine functions, including the generalization to directions lying outside the first quadrant (reviewable by reading (or re-reading) Section (c) of this chapter (Chapter 3) of the course)*
- *Familiarity with using the generalized definitions of the sine and cosine, together with the algebraic expression for the secant, cosecant, tangent and cotangent in terms of the sine and cosine, to generalize the secant, cosecant, tangent and cotangent functions to non-first-quadrant angles (also reviewable by reading (or re-reading) Section (c) of this chapter of the course)*

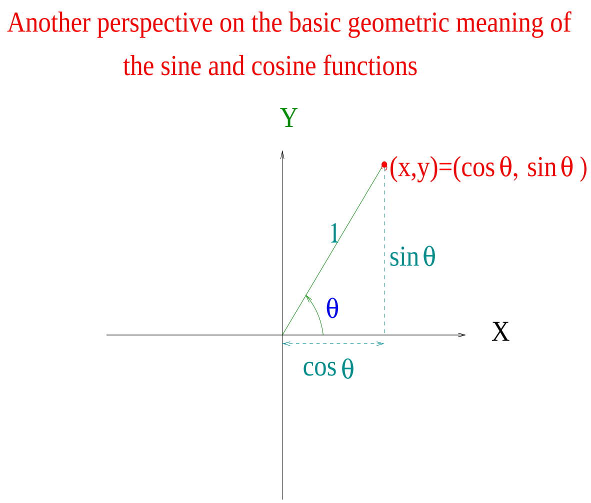
In this section of the course we will consider an alternate way of thinking about the trig functions of angles in the second, third and fourth quadrants which still involves right angle triangles, and which, as a result, some students find easier to think about since it allows the trig functions of such angles to still be defined using the basic geometrical definitions.

This alternate "right-angle triangle based" perspective will prove useful for working out the trig function values for new "special angles", corresponding to directions one-third, one-half and two-thirds of the way through any of Quadrants II, III, or IV (a topic to be discussed in detail in the next section (Section (e)) of this chapter of the course)

(Angles corresponding to the directions along the $\pm x$ or $\pm y$ axes, however, turn out not be associated in any natural way with any right-angle triangle, making the unit circle picture far and away the easiest perspective to use in working out the values of the trig functions for these directions.)

- The first step in our outline of the basic argument is to recall that, for any first quadrant angle θ, there is a right-angle triangle naturally associated with it which lies in the first quadrant and has its hypoteneuse pointing in the θ direction (as shown in Figure 1).

FIG. 1:



- The next step in the argument will be to show that, by imagining taking this first quadrant triangle and “flopping it over” appropriately into either the second, third or fourth quadrant, we obtain “new” triangles (“new” in the sense that their hypoteneuses point in new (Quadrants II, III and IV) directions). These new triangles, being flopped over copies of the original triangle, have a special relation to the original first quadrant triangle. (Figures 2, 3 and 4 below show the exact geometrical relation of these new triangles to the original first quadrant triangle, and will be discussed in more detail in what follows.)

- By labelling the sides of these new triangles in a way that is compatible with our generalized unit circle picture for the sine and cosine functions, we will see that we can use these new triangles to help us create generalized versions of the geometrical definitions of the six trig functions as ratios of sides of right angle triangles which work in all four quadrants, and not just in Quadrant I.

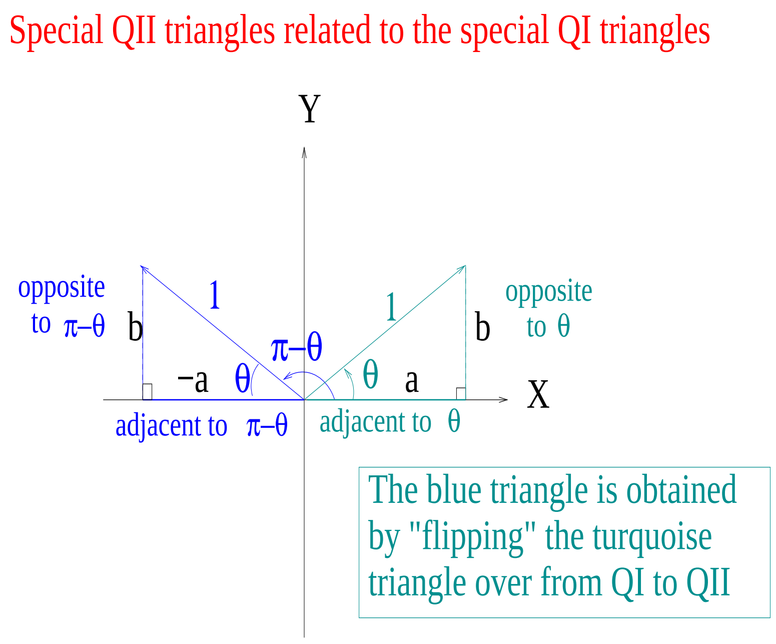
- We now go through the argument just outlined in more detail, beginning with the figures showing the second, third and fourth quadrant triangles obtained by flopping over the original first quadrant triangle.

- Note that the figures show both the angle θ specifying the direction of the hypoteneuse of the original first quadrant triangle and the new angle specifying the the direction of the new second, third or fourth quadrant triangle. The explanations for the values of these new angles in terms of θ are given after the figures.

THE RELATED "FLOPPED-OVER" SECOND, THIRD AND FOURTH QUADRANT TRIANGLES

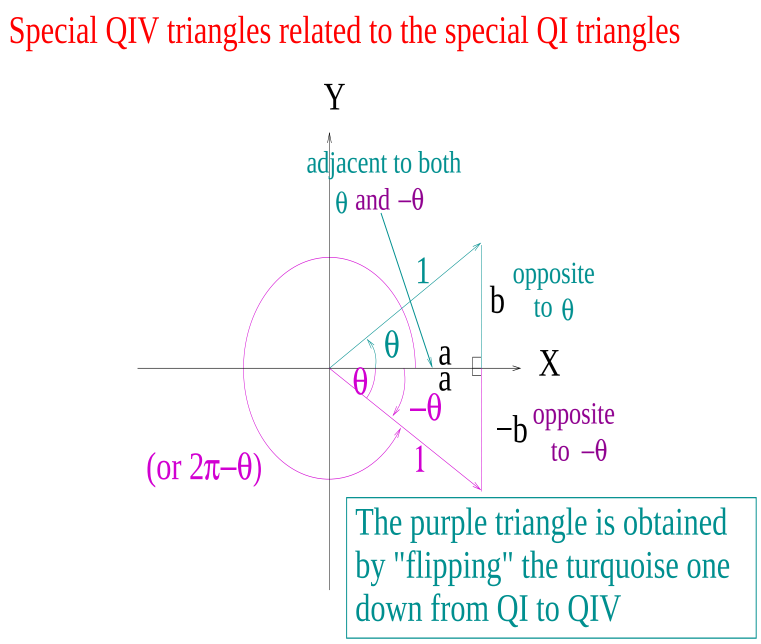
- If one takes the first quadrant triangle and "flops it over to the left" into the second quadrant, one has a "new" right triangle whose hypoteneuse points in a new (in this case, second quadrant) direction. This is illustrated in Figure 2.

FIG. 2:



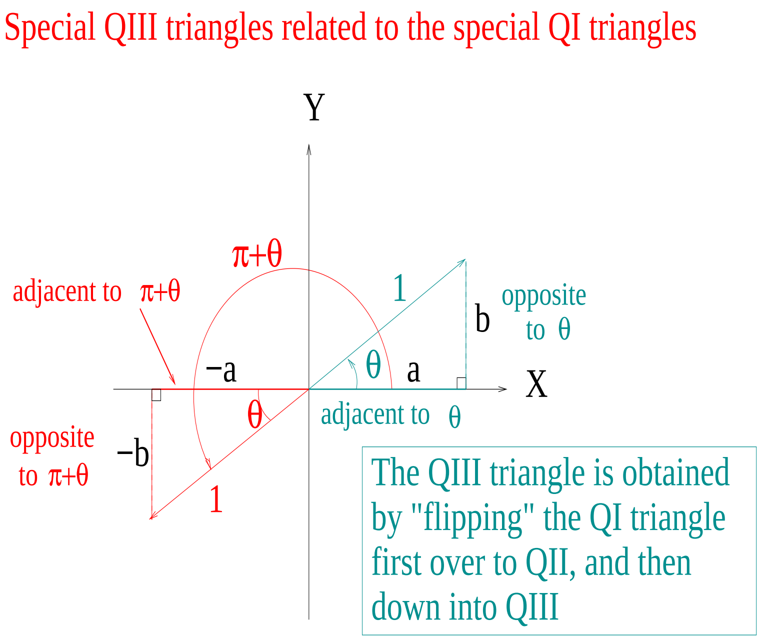
- If one takes the first quadrant triangle and "flops it over", this time down into the fourth quadrant, one has a different "new" right triangle, this time with its hypoteneuse pointing in a fourth quadrant direction. This is illustrated in Figure 3.

FIG. 3:



- Finally, if one takes the first quadrant triangle and "flops it over", first to the left, into the second quadrant, and then, after that, down into the third quadrant, one has yet another "new" right triangle, this time with its hypoteneuse pointing in a third quadrant direction. This is illustrated in Figure 4.

FIG. 4:



- The relations between the angles describing the directions in which the hypoteneuses of the three new triangles point and the angle θ specifying the direction of the original first quadrant hypoteneuse is straightforward to work out geometrically (and is in fact already shown in the three figures).
  - In Figure 2, the sum of the angle describing the new Quadrant II hypoteneuse direction and the internal angle θ is π radians (corresponding to half of a full rotation), which implies that the angle describing the direction of the hypoteneuse is $\pi - \theta$, as shown in Figure 2.
  - Similarly, in Figure 4, starting at the 0 angular position (the $+x$ axis direction) and rotating half of a full revolution in the positive (counterclockwise) direction gets us to the $-x$ axis direction. To get the the third quadrant direction which characterizes the hypoteneuse of the new third quadrant triangle, we have to rotate further in the counterclockwise direction by an amount θ (the internal angle of the triangle). The hypoteneuse of the third quadrant "double-flopped" triangle thus points in the $\pi + \theta$ direction, as shown in Figure 4.
  - You should check that you are able to give a geometrical argument to explain why the direction of the hypoteneuse of the fourth quadrant triangle shown in Figure 3 can be characterized by the angle $-\theta$ (or, equivalently, $2\pi - \theta$).

- It is easy to use the unit circle picture for the sine and cosine to find out the relation of the sines and cosines of the “new” second, third and fourth quadrant angles in terms of the sine and cosine of the original first quadrant direction.

- To do this, we start by imagining the original first quadrant triangle has hypoteneuse of length *1*, so that
  - the x and y coordinates of its tip, which we denote by $x = a$ and $y = b$ in the three figures above, are given by $a = cos(\theta)$ and $b = sin(\theta)$,
  - equivalently, $a = cos(\theta)$ is the length of the horizontal side of the first quadrant triangle while $b = sin(\theta)$ is the length of its vertical side.

- The rest of the argument will rely on the facts that
  - the coordinates of the tip of the hypoteneuse of the original first quadrant triangle are then $(x, y) = (cos(\theta)\,,\, sin(\theta))$;
  - the coordinates of the tip of the hypoteneuse of the new triangle are, similarly, $(x', y') = (cos(\theta')\,,\, sin(\theta'))$ where θ' is the angle describing the direction of the hypoteneuse of the new second, third or fourth quadrant triangle;
  - because the new triangles are obtained by flopping over the original triangle without changing the sizes of any of the sides, the coordinates (x', y') of the tip of each of the new hypoteneuses are easily written down in terms of the coordinates (x, y) of the tip of the original hypoteneuse.

- In what follows, we'll go through the argument in detail for the second quadrant case, and then outline more quickly the third and fourth quadrant cases.

- The detailed argument for the second quadrant case:

  When the first quadrant triangle is flopped over from the right side of the y axis $(x > 0)$ to the left side $(x < 0)$, to produce the second quadrant triangle of Figure 2 (whose hypoteneuse points in the second quadrant direction characterized by angle $\phi = \pi - \theta$),

  – The x coordinate of the tip changes sign, while the y coordinate does not.

  – The unit circle picture for the sine and cosine then tells us that the new x-coordinate, $x = -a$ is equal to $cos(\phi)$ and hence that $cos(\phi) = cos(\pi - \theta) = -a = -cos(\theta)$.

  – Similarly, the new y-coordinate, $y = b$ is equal to $sin(\phi)$, and hence $sin(\phi) = sin(\pi - \theta) = b = sin(\theta)$.

  – Since the new negative x coordinate, $x = -a$, is just the location in x of the right-angle end of the base of the second quadrant triangle, if we were to treat the horizontal side of the new triangle as being the side "adjacent" to the new angle ϕ and, in addition, label it as having the new "negative length" $-a$, the geometrical first quadrant rule that the cosine of an angle θ is the ratio of the adjacent side to the hypoteneuse of the triangle associated with θ would work to give the correct result for the cosine of $\phi = \pi - \theta$, $cos(\phi) = adjacent/hypoteneuse = (-a)/1 = -a$, even though ϕ is now a second quadrant, not a first quadrant angle.

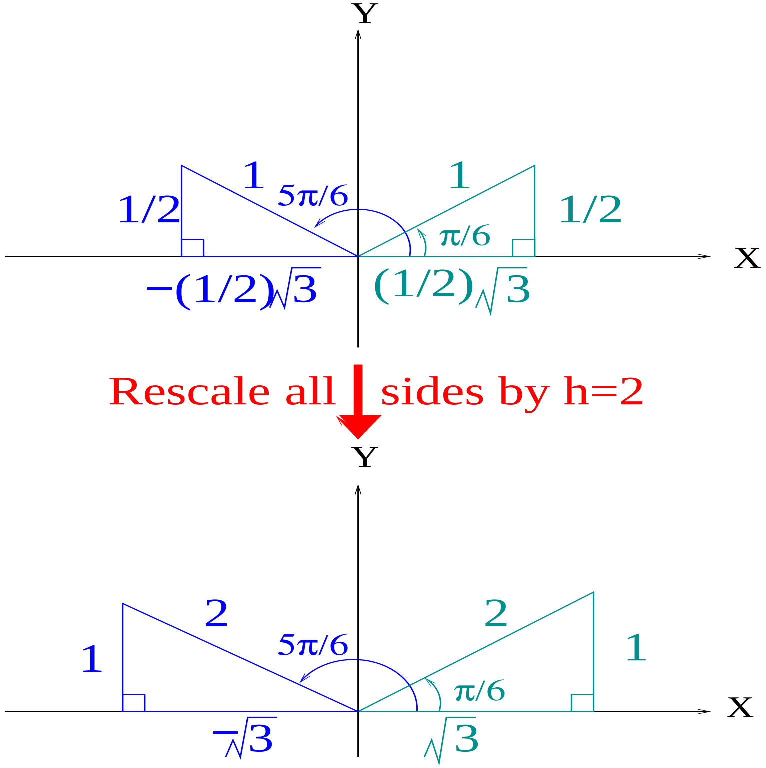
- Similarly, since the new positive y coordinate, $y = b$, is just the length of the vertical side of the new second quadrant triangle, if we were to treat the vertical side of the new triangle as being the side "opposite" to the new angle ϕ and, in addition, label it as having the new length $+b$, the geometrical first quadrant rule that the sine of an angle ϕ is the ratio of the opposite side to the hypoteneuse of the triangle associated with ϕ would work to give the correct result for the sine of $\phi = \pi - \theta$, $sin(\phi) = opposite/hypoteneuse = b/1 = b$, even though ϕ is a second quadrant, not a first quadrant angle.

- Once we know that the new geometrical rules give us the right values for the sine and cosine of $\phi = \pi - \theta$, the fact that the algebraic relations for the other functions in terms of the sine and cosine will be used to define the other four trig functions of ϕ ensures that our old first quadrant geometric ratio definitions will also work to give us the correct answers for the secant, cosecant, tangent and cotangent of ϕ (some explicit examples will be worked out in the next section of this chapter).

- For the special first quadrant angles $\theta = \pi/6$, $\pi/4$ and $\pi/3$, the representative triangles we constructed in Chapter 2 to help us generate the trig functions of these angles did not have hypoteneuses of length *1*. In such cases it may be convenient to rescale all sides of both the first and second quadrant triangles by the same common factor so that the hypoteneuses of both triangles will become whatever we would like them to be. Such a common factor will cancel out when we take ratios of sides, so either the original figure, or the rescaled version, can be used in the geometrical "ratio-of-sides" definitions.

– An illustration of this last point is provided in Figure 5 for the case $\theta = \pi/6$. (The accompanying detailed explanation is also given below.)

FIG. 5:



∗ In the upper half of the figure is a version of a first quadrant $\pi/6$ triangle with hypoteneuse of length *1* and the flopped-over second quadrant related triangle, also with hypoteneuse of length *1*. The lengths of the sides of the triangles are labelled as per the discussion above, using the information from Chapter 2 that $sin(\pi/6) = 1/2$ and $cos(\pi/6) = \frac{1}{2}\sqrt{3}$.

* The first quadrant triangle, however, is not the version of a $\pi/6$ triangle we constructed in Chapter 2 (the Chapter 2 version had sides 2, $\sqrt{3}$ and 1 and hence was twice as big as that shown in the upper half of the figure here).

* We can, however, switch back to the version of the $\pi/6$ triangle we are familiar with from Chapter 2 by rescaling all the sides of both triangles by the common factor 2. This yields the rescaled pair of triangles shown in the bottom half of Figure 5, in which the first quadrant triangle is now the one we are familiar with from Chapter 2.

* The rescaled second quadrant triangle in the bottom half of the figure is now still just the flopped-over version of the larger first quadrant triangle in the bottom half of the figure.

* Since the extra common factors of 2 cancel when we take ratios, the ratios of the sides of the lower second quadrant triangle are the same as the corresponding ratios of the sides of the upper second quadrant triangle.

* Thus, so long as we remember that the vertical side is to be interpreted as having positive length and being opposite to $\pi - (\pi/6) = 5\pi/6$ and the horizontal side as having negative "length" and being adjacent to $5\pi/6$, we can use either the upper or the lower second quadrant triangle to compute the trig functions of $5\pi/6$.

* Three of the resulting six trig functions are worked out below as examples (in each case two calculations are given, one using the upper triangle and one using the lower triangle)

$$
\begin{aligned}
sin(5\pi/6) &= opposite/hypoteneuse \\
&= (1/2)/1 = 1/2 \ (upper\ triangle) \\
&= 1/2 = 1/2 \ (lower\ triangle)
\end{aligned}
$$

$$
\begin{aligned}
cos(5\pi/6) &= adjacent/hypoteneuse \\
&= (-\frac{1}{2}\sqrt{3})/1 = -\frac{1}{2}\sqrt{3} \ (upper\ triangle) \\
&= (-\sqrt{3})/2 = -\frac{1}{2}\sqrt{3} \ (lower\ triangle)
\end{aligned}
$$

$$
\begin{aligned}
tan(5\pi/6) &= opposite/adjacent \\
&= \frac{1/2}{-(\sqrt{3})/2} = -1/\sqrt{3} \ (upper\ triangle) \\
&= \frac{1}{-\sqrt{3}} = -1/\sqrt{3} \ (lower\ triangle)
\end{aligned}
$$

* We can see, especially in the case of the tangent, that it is convenient to be able to use the rescaled lower triangle rather than the unrescaled upper one (and hence avoid division involving fractions) even though the two of course give the same final answer.

- The fourth quadrant case:

  In an analogous way, for the case when a first quadrant triangle is flopped over from above the x-axis $(y > 0)$ to below the x-axis $(y < 0)$, so one has a new triangle whose hypoteneuse points in the fourth quadrant direction characterized by the new angle $\phi = -\theta$, as in Figure 3:

  - The y coordinate of the tip changes sign from b to -b, while the x coordinate remains unchanged, and equal to a.

  - The unit circle picture for the sine and cosine then tells us that the new x-coordinate, $x = a$ is equal to $cos(\phi)$ and hence that $cos(\phi) = a = cos(\theta)$.

  - The new y-coordinate, $y = -b$ is equal to $sin(\phi)$, and hence $sin(\phi) = -b = -sin(\theta)$.

  - We can obtain these same correct results for the sine and cosine of the fourth quadrant angle $\phi = -\theta$ if we

    * treat the horizontal side of the fourth quadrant triangle as being "adjacent" to the angle ϕ and the vertical side as being "opposite" to ϕ;

    * label the resulting rightward-pointing adjacent (horizontal) side with the positive length label a and the resulting downward-pointing (i.e., into negative y territory) opposite (vertical) side with the "negative length" label -b; and

    * use the standard geometrical definitions familiar from the first quadrant (with the sine being the ratio of the opposite side to the hypoteneuse and the cosine being the ratio of the adjacent side to the hypoteneuse) to also define the sine and cosine of the new fourth quadrant angle $\phi = -\theta$.

- The standard geometrical definitions familiar from the first quadrant (in which the cosecant is the ratio of the hypoteneuse to the opposite side, the secant is the ratio of the hypoteneuse to the adjacent side, the tangent is the ratio of the opposite to the adjacent side, and the cotangent is the ratio of the adjacent to the opposite side) then also give the correct values for the cosecant, secant, tangent and cotangent of the fourth quadrant angle $\phi = -\theta$.

- As for the second quadrant case, we can rescale all sides of both the first and the fourth quadrant triangles by the same amount, to create triangles with hypoteneuses having lengths different from *1*; the geometrical ratio definitions still yield the correct results for all six trig function of the fourth quadrant angle $-\theta$ provided one interprets the downward-pointing vertical side of the fourth quadrant triangle as being opposite to ϕ and having negative length -*b* and the rightward-pointing horizontal side as being adjacent to ϕ and having positive length *a*.

- The third quadrant case: For the case of the third quadrant triangle of Figure 4, associated with the third quadrant angle $\phi = \pi + \theta$, and obtained by taking the first quadrant triangle and double-flopping it, first from to the right of the *y*-axis $(x > 0)$ to the left of the *y*-axis $(x < 0)$, and then from above the x-axis $(y > 0)$ to below the x-axis $(y < 0)$,
  - The y coordinate of the tip changes sign from *b* to -*b*. The x coordinate also changes sign, from *a* to it -a.
  - The unit circle picture for the sine and cosine then tells us that the new *x*-coordinate, $x = -a$ is equal to $cos(\phi)$ and hence that $cos(\phi) = cos(\pi + \theta) = -a = -cos(\theta)$.
  - The new *y*-coordinate, $y = -b$ equals $sin(\phi)$, and hence $sin(\phi) = sin(\pi + \theta) = -b = -sin(\theta)$.

- We can obtain these same correct results for the sine and cosine of the third quadrant angle $\phi = \pi + \theta$ if we
  * treat the horizontal side of the third quadrant triangle as being "adjacent" to the angle ϕ and the vertical side as being "opposite" to ϕ;
  * label the resulting leftward-pointing (i.e., into negative x territory) adjacent (horizontal) side with the negative length label -a and the resulting downward-pointing (i.e., into negative y territory) opposite (vertical) side with the negative length label -b; and
  * use the standard geometrical definitions familiar from the first quadrant (with the sine being the ratio of the opposite side to the hypoteneuse and the cosine being the ratio of the adjacent side to the hypoteneuse) to also define the sine and cosine of the new third quadrant angle $\phi = \pi + \theta$
- The standard geometrical definitions, familiar from the first quadrant and already noted above, then also give the correct values for the cosecant, secant, tangent and cotangent of the third quadrant angle $\phi = \pi + \theta$.
- As for the cases above, we can rescale all sides of both the first and the third quadrant triangles by the same amount, to create triangles with hypoteneuses having lengths different from *1*; the geometrical ratio definitions still yield the correct results for all six trig function of the third quadrant angle $\phi = \pi + \theta$ provided one interprets the downward-pointing vertical side of the third quadrant triangle as being opposite to ϕ and having negative length -b and the leftward-pointing horizontal side as being adjacent to ϕ and having negative length -a.

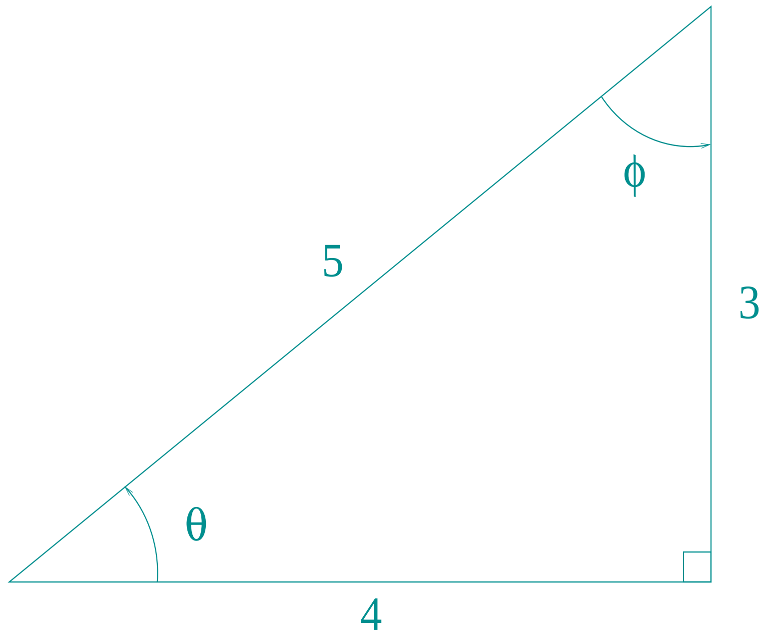
The above geometrical constructions can be summarized in the following manner, yielding a geometrical picture of the trig functions which works in all four quadrants:

Given a right triangle with non-right Quadrant I angle θ, hypoteneuse h, adjacent side a and opposite side b, one can obtain the trig functions of (i) the related Quadrant II angle $\phi = \pi - \theta$, (ii) the related Quadrant III angle $\phi = \pi + \theta$ and (iii) the related Quadrant IV angle $\phi = -\theta$ by

- *flopping the original first quadrant triangle over into the desired final quadrant;*
- *associating the new triangle to the relevant new angle ϕ (which specifies the direction in which the new hypoteneuse points);*
- *treating the horizontal side of the resulting new triangle as being "adjacent" to the new angle and the vertical side as being "opposite" to the new angle;*
- *labelling the hypoteneuse of the new triangle with the same (positive) length as used for the original first quadrant triangle;*
- *changing the sign of the "length" of the adjacent side from positive to negative if the horizontal side of the new triangle extends to the left of the y-axis and the sign of the "length" of the opposite side from positive to negative if the vertical side of the new triangle extends below the x-axis;*
- *defining the trig functions of the new second, third, or fourth quadrant angle ϕ in the standard geometrical way as ratios of the new hypoteneuse, adjacent and opposite sides.*

The following examples, connected to Figure 6, provides a concrete illustration of the points discussed above. (Note that the angle labelled θ in the figure corresponds to that labelled θ in the first quadrant triangle in all of the discussions above.) The numerical value of θ here is not really needed, but for completeness I note that it is approximately $0.204833\,\pi$ radians.

FIG. 6:



- In terms of the notation above, the hypoteneuse of the original first quadrant triangle is $h = 5$, the side adjacent to θ (the horizontal side) is $a = 4$ and the side opposite to θ (the vertical side) is $b = 3$.

- The related Quadrant II triangle thus corresponds to the rescaled version of Figure 2 with hypoteneuse labelled by *5*, the side adjacent to $\pi - \theta$ (the horizontal side) labelled by *-4* and the side opposite to $\pi - \theta$ (the vertical side) labelled by *3*. It then immediately follows that

$$\begin{aligned}
sin(\pi - \theta) &= opposite/hypoteneuse = 3/5 \\
cos(\pi - \theta) &= adjacent/hypoteneuse = (-4)/5 = -4/5 \\
tan(\pi - \theta) &= opposite/adjacent = 3/(-4) = -3/4 \\
csc(\pi - \theta) &= hypoteneuse/opposite = 5/3 \\
sec(\pi - \theta) &= hypoteneuse/adjacent = 5/(-4) = -5/4 \\
cot(\pi - \theta) &= adjacent/opposite = (-4)/3 = -4/3
\end{aligned}$$

- Similarly related Quadrant III triangle corresponds to the rescaled version of Figure 4 with hypoteneuse labelled by *5*, the side adjacent to $\pi + \theta$ (the horizontal side) labelled by *-4* and the side opposite to $\pi + \theta$ (the vertical side) labelled by *-3*. It then immediately follows that

$$\begin{aligned}
sin(\pi + \theta) &= opposite/hypoteneuse = (-3)/5 = -3/5 \\
cos(\pi + \theta) &= adjacent/hypoteneuse = (-4)/5 = -4/5 \\
tan(\pi + \theta) &= opposite/adjacent = (-3)/(-4) = 3/4 \\
csc(\pi + \theta) &= hypoteneuse/opposite = 5/(-3) = -5/3 \\
sec(\pi + \theta) &= hypoteneuse/adjacent = 5/(-4) = -5/4 \\
cot(\pi + \theta) &= adjacent/opposite = (-4)/(-3) = 4/3
\end{aligned}$$

- The case of the related Quadrant IV triangle is left as an exercise for you to work out below.

EXERCISES: *(All angles below are understood to be given in radians)*

1. With θ the angle shown in Figure 6 above, work out the values of all six trig functions of the related fourth quadrant angle $\phi = -\theta$.

2. With α the first quadrant angles shown in Figure 7, determine the values of all six trig functions of the related third quadrant angle $\pi + \alpha$

3. With β the first quadrant angles shown in Figure 7, determine the values of all six trig functions of the related second quadrant angle $\pi - \beta$ and the related fourth quadrant angle $-\beta$. (It will make your life a lot easier if you first reorient the triangle so that β lies in the usual position, corresponding to θ in the discussions above.)

FIG. 7: