“IDENTITIES” VERSUS “EQUATIONS”

Students are sometimes confused about the distinction between identities and equations. This happens most often for students who have not had the idea behind an “identity” fully explained to them before. In this section of the course we will aim to make this distinction clear, since understanding what is meant by an identity will be crucial to understanding the rest of the material in this chapter of the course.

The basic idea defining the meaning of the term “identity” is the following:

Identities are equations which are true “for all values of the variable(s)”, i.e., equations which are true for all values of the variable(s) for which both sides of the equation defined.

Identities thus form a subset of the set of all equations.

Let’s look at a couple of examples to make sure the underlying idea is clear.

- Example 1:
  - Consider the two equations

    $$(1 - x^2) + x^2 = 1 \quad and \quad (1 - x^2) - x^2 = 1$$

    which look very similar since they differ only by the change of the plus sign in the first equation to a minus sign in the second equation.
  - The first equation, however, is true to all real numbers x since the $-x^2$ from the first term cancels the $+x^2$ from the second term. The first equation is thus what we call an identity (we will see less trivial identities later in this chapter).

– In contrast, the second equation is equivalent to $1 - 2x^2 = 1$, which is equivalent to $-2x^2 = 0$ and hence is only true for $x = 0$.

– The second equation is thus NOT an identity since both sides of the equation are defined for all real numbers x, but the equation is satisfied for only one of these numbers, $x = 0$.

– In the case like that of the second equation, which is not an identity, we call the values of the variable(s) for which the equation IS satisfied the "solutions of the equation". Here, $x = 0$ happens to be the only solution of the second equation.

- Example 2:

– Consider the two equations

$$\frac{1}{\sqrt{x^2+1}-1} - \frac{1}{\sqrt{x^2+1}+1} = \frac{2}{x^2} \quad and \quad \frac{1}{\sqrt{x^2+1}-1} - \frac{1}{\sqrt{x^2+1}+1} = \frac{1}{2}$$

which have the same left hand sides, but slightly different right hand sides.

– The left hand side of the first equation, however, can be re-written

$$\begin{aligned}\frac{\sqrt{x^2+1}+1}{\left(\sqrt{x^2+1}+1\right)\left(\sqrt{x^2+1}-1\right)} - \frac{\sqrt{x^2+1}-1}{\left(\sqrt{x^2+1}-1\right)\left(\sqrt{x^2+1}+1\right)} &= \frac{\left(\sqrt{x^2+1}+1\right) - \left(\sqrt{x^2+1}-1\right)}{\left(\sqrt{x^2+1}-1\right)\left(\sqrt{x^2+1}+1\right)} \\ &= \frac{2}{\left[\sqrt{x^2+1}\right]^2 - 1} = \frac{2}{x^2}\end{aligned}$$

which is equal to the right hand side of the equation, making the equation true for all real x (except $x = 0$, for which neither side of the equation is defined).

- The same manipulation to the left hand side of the second equation cnverts the second equation into the equivalent equation $2/x^2 = 1/2$, which is itself equivalent, after cross-multiplication, to $4 = x^2$, which is satisfied only for $x = \pm 2$.

- Since the first equation is true for all real x for which both sides of the equation are defined, it is an identity.

- Since the second equation (whose two sides are also defined for all real x except for $x = 0$) is, in contrast, only satisfied for the two values, $x = \pm 2$, the second equation is NOT an identity (in this case it has two solutions).

- Example 3:

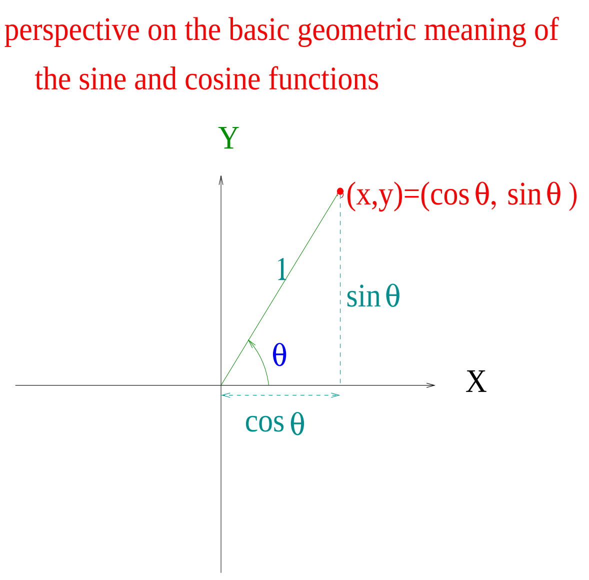
  - Below I reproduce the figure showing the unit circle picture for the sine and cosine, introduced earlier in Section (b) of Chapter 3 of the course.
  - Applying Pythagoras' Theorem to the triangle shown in the figure, remembering that the hypoteneuse in this case has length 1, we see that $[sin(\theta)]^2 + [cos(\theta)]^2 = 1^2 = 1$. (This is usually written in the more compact form $sin^2(\theta) + cos^2(\theta) = 1^2 = 1$.

  - Consider this equation and the very similar equation $sin^2(\theta) - cos^2(\theta) = 1^2 = 1$.

    * As we just saw, the first equation follows from Pythagoras' Theorem and hence is true for all θ. It is thus an identity.

    * In contrast, if we use the Pythagoras' Theorem result, in the alternate form $cos^2(\theta) = 1 - sin^2(\theta)$, we can rewrite the left hand side of the second equation as $sin^2(\theta) - [1 - sin^2(\theta)] = 2\,sin^2(\theta) - 1$.

FIG. 1:



* This allows us to rewrite the second equation as

$$2\,sin^2(\theta)\;-\;1=1\;\Rightarrow\;2\,sin^2(\theta)\;=\;2\;\Rightarrow\;sin^2(\theta)\;=\;1\;\Rightarrow\;sin(\theta)\;=\;\pm 1.$$

* The only θ for which $sin(\theta)\;=\;\pm 1$ are, however, $\theta=\pm\frac{\pi}{2}+2\pi N$, for any $N=0,\,\pm 1,\,\pm 2,\,\cdots$. Even though, in this case, there are an infinite number of solutions, the original second equation is NOT satisfied for ALL θ and so, again, the second equation is NOT an identity.