THREE TRIG IDENTITIES BASED ON PYTHAGORAS' THEOREM

In this section of the course, we derive , and discuss the use of, three trig identities which are a direct consequence of applying Pythagoras' Theorem, taking into account the basic geometric definitions of the sine and cosine functions.

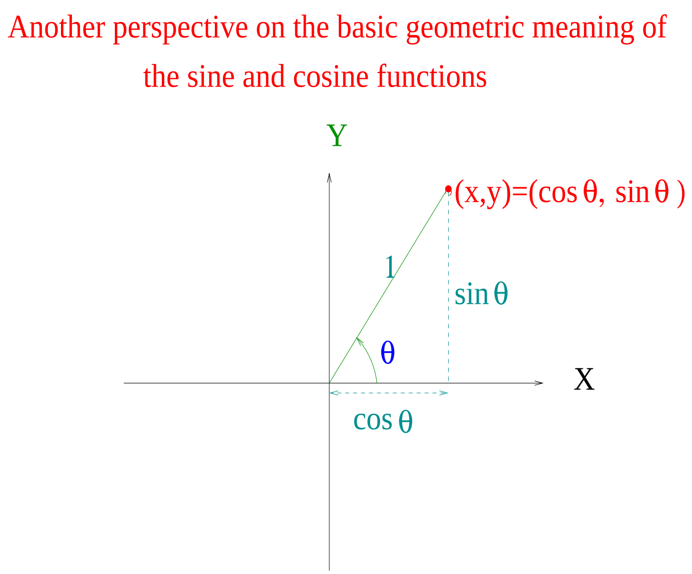
The main pieces of background information required to follow the material in this section of the course are

- *Pythagoras' Theorem (see Section (b) of Chapter 2 of the course for a detailed geometrical proof);*
- *the basic geometrical (right-triangle-based) definitions of the sine and cosine (discussed in Section (c) of Chapter 2 of the course);*
- *the algebraic relations expressing the tangent, cotangent, secant and cosecant as ratios involving the sine and cosine (discussed in Section (a) of Chapter 3 of the course); and*
- *the unit circle picture for generalizing the sine and cosine functions, and from these, also the tangent, cotangent, secant and cosecant functions, from Quadrant I to Quadrant II, III and IV values of the relevant angle (discussed in Sections (b) and (c) of Chapter 3 of the course).*

THE PYTHAGORAS' THEOREM BASED IDENTITIES

- The basic identity $sin^2(\theta) + cos^2(\theta) = 1$:
  - Figure 1 provides a reminder of the unit circle picture of the sine and cosine functions, whose origin was discussed in detail in Sections (b) and (c) of Chapter 3 of the course.

FIG. 1:



  - Pythagoras' Theorem states that, in general, for a right-angle triangle having hypoteneuse h, and non-hypoteneuse sides a and b, $h^2 = a^2 + b^2$.

– Applying Pythagoras' Theorem to the triangle above, where we have $h = 1$, and can let $a = sin(\theta)$ and $b = cos(\theta)$, we obtain the first of our Pythagoras' Theorem based identities,

$$sin^2(\theta) + cos^2(\theta) = 1 \tag{1}$$

where $sin^2(\theta)$ and $cos^2(\theta)$ are standard shorthand for $[sin(\theta)]^2$ and $[cos(\theta)]^2$ respectively.

– Some students have been required to memorize this identity in high school without being told that it is a consequence of Pythagoras' Theorem! Knowing that it comes from Pythagoras' Theorem makes it much easier to remember that the left hand side involves the squares of the sine and cosine functions.

- The related identities $sec^2(\theta) = tan^2(\theta) + 1$ and $csc^2(\theta) = 1 + cot^2(\theta)$:

– Starting from the basic identity given in Equation 1, if we divide both sides of the equation by $cos^2(\theta)$, we find

$$\frac{sin^2(\theta)}{cos^2(\theta)} + 1 = \frac{1}{cos^2(\theta)} \Rightarrow \left[\frac{sin(\theta)}{cos(\theta)}\right]^2 + 1 = \left[\frac{1}{cos(\theta)}\right]^2 \Rightarrow tan^2(\theta) + 1 = sec^2(\theta)$$

(using the algebraic relations $sec(\theta) = 1/cos(\theta)$ and $tan(\theta) = sin(\theta)/cos(\theta)$ in the last step).

– Similarly, starting from the basic identity given in Equation 1, if we divide both sides of the equation by $sin^2(\theta)$, we find

$$1 + \frac{cos^2(\theta)}{sin^2(\theta)} = \frac{1}{sin^2(\theta)} \Rightarrow 1 + \left[\frac{cos(\theta)}{sin(\theta)}\right]^2 = \left[\frac{1}{sin(\theta)}\right]^2 \Rightarrow 1 + cot^2(\theta) = csc^2(\theta)$$

(using the algebraic relations $csc(\theta) = 1/sin(\theta)$ and $cot(\theta) = cos(\theta)/sin(\theta)$ in the last step).

- Summarizing, we have the two new identities, also based on Pythagoras' Theorem, though in a form where this connection is not so immediately obvious as it was for the earlier identity,

$$\begin{aligned} tan^2(\theta) + 1 &= sec^2(\theta) \\ 1 + cot^2(\theta) &= csc^2(\theta) \end{aligned} \tag{2}$$

- Note that, since the sine and cosine are defined for all angles, both sides of the Equation 1, are defined for all θ. Since we have shown that the equation is satisfied for all θ it is an identity.

- In the first of Equations 2, in contrast, when θ corresponds to a direction along either the $+y$ or $-y$ axis, both the $tan^2(\theta)$ factor on the left hand side and the $sec^2(\theta)$ factor on the right hand side are undefined and neither side of the equation makes sense. For all other angles, however, we have shown that equation is satisfied, and hence that it is an identity.

- Similarly, when θ corresponds to a direction along either the $+x$ or $-x$ axis, both the $cot^2(\theta)$ factor on the left hand side and the $csc^2(\theta)$ factor on the right hand side of the second of Equations 2 are undefined, and neither side of the equation makes sense. For all other angles, however, we have shown that the second equation is satisfied, and hence that it is also an identity.

USES OF THE PYTHAGORAS' THEOREM IDENTITIES

- One obvious use of the Pythagoras' Theorem based identities is "substitutional". That is, we can always replace $sin^2(x) + cos^2(x)$ by 1, always replace $tan^2(x) + 1$ by $sec^2(x)$, and always replace $cot^2(x) + 1$ by $csc^2(x)$ (or vice versa, if that is more convenient for some reason). This is the most common way that students use these identities.

- These identities, however, also contain some additional useful information about the relations between the values of the pair of trig functions occurring in the same identity. Examples 1 through 6 below illustrate what I mean by this statement. illustrate what I mean by this statement:

  - Example 1:

    * Suppose we are told that we have an angle θ which is such that $sin(\theta) = 1/3$. (This is not one of the values corresponding to any of the special directions we discussed earlier, so we don't know the exact values of the θ which satisfy this condition, only that, since the sine value is positive, all such θ have to correspond to directions in either the first or second quadrant.)

    * We can then use Eq. (1) to conclude that

$$cos^2(\theta) = 1 - \frac{1}{9} = \frac{8}{9} \;\Rightarrow\; cos(\theta) \;=\; \pm\sqrt{\frac{8}{9}} \;=\; \pm\frac{2\sqrt{2}}{3}\ .$$

    * Note that the identity knows about both the first and second quadrant possibilities for θ. If θ is in the first quadrant, then $cos(\theta)$ has to be positive, and the positive square root will be the correct choice of solution. In contrast, if θ is in the second quadrant, then $cos(\theta)$ will be negative, and the negative square root will be the correct choice of solution.

* Note also that, if we are given not just the value $sin(\theta) = 1/3$, but also told which of the two possibilities (first or second quadrant) θ corresponds to, we will then know both $sin(\theta)$ and $cos(\theta)$, and hence the values of ALL of the other four trig functions as well.

– Example 2:

* Similarly, if we were given $sin(\theta) = -1/3$ (so θ corresponds to either a third or fourth quadrant direction) we would also find that $cos(\theta) = \pm\frac{2\sqrt{2}}{3}$, with the positive square root corresponding to the fourth quadrant possibility and the negative square root to the third quadrant possibility.

* Again, if we are given not just the initial value $sin(\theta) = -1/3$ but are also told which of the two possibilities (third or fourth quadrant) θ corresponds to, we will then know which of the two possible signs to choose for $cos(\theta)$, and, at that point, knowing both $sin(\theta)$ and $cos(\theta)$, will also know the values of ALL of the other four trig functions as well.

– *In both Example 1 and Example 2, once we are given the value of the sine AND the information about which of the two possible quadrants is the one of interest, we immediately also determine the cosine, and hence the values of all six trig functions for the angle in question. An analogous statement would be true is we started with a value for the cosine, together with a statement about which of the two possible quadrants that value might correspond to is the one we're interested in, and used the identity to work out the sine of that angle.*

– Example 3:

* Suppose were are told that $tan(\theta) = -5/2$ (a possible value for the tangent). Since this is negative, θ must correspond to either a second or fourth quadrant direction, though we don't know the exact values of such θ since $-5/2$ is not one of the values of the tangent for the special directions we studied earlier.

* The first of the identities in Equations 2 then implies

$$sec^2(\theta) = \left(-\frac{5}{2}\right)^2 + 1 = \frac{29}{4} \;\Rightarrow\; sec(\theta) \;=\; \pm\frac{\sqrt{29}}{2}$$

  with the positive square root corresponding to the fourth quadrant possibility and the negative square root to the second quadrant possibility (the secant is positive in the fourth quadrant and negative in the second quadrant).

* In this example, if we are given not only the value of the tangent, but also are told which of the two possibilities (second or the fourth quadrant) this value corresponds to

  · We will then also be able to completely determine the value of the secant.

  · Then, since $cos(\theta) = 1/sec(\theta)$ and $sin(\theta) = tan(\theta)\,cos(\theta) = tan(\theta)/sec(\theta)$, the values of the tangent and secant can be used to also get the values of the sine and cosine. With these values, we can then also work out the values of the remaining two trig functions, $csc(\theta) = 1/sin(\theta)$ and $cot(\theta) = cos(\theta)/sin(\theta)$.

* *Once again, the value of one of the trig functions in the identity, together with the additional information about which of the two possible quadrants corresponding to that value is the one of interest, allows us to work out the values of all six trig functions for the direction corresponding to the angle in question.*

– Example 4:

* Suppose we think that we have an angle θ such that $csc(\theta) = 1/2$ (having forgotten for the moment that $1/2$ is not a possible value for the cosecant). If we then try to use the second of the identities in Equations 2 to solve for the cotangent of the same angle, we find

$$cot^2(\theta) = csc^2(\theta) - 1 = \left(\frac{1}{2}\right)^2 - 1 = -\frac{3}{4} .$$

* But of course the left hand side of this equation is a square and hence ≥ 0, whereas the right hand side is < 0, so in fact the identity tells us that no such θ can exist.

* Even though we "forgot" about the restriction on the values of the cosecant, the identity "knew" about it, and reminded us of this fact.

– Example 5:

* Let's instead take an example with an allowed value of the cosecant, $csc(\theta) = -3$ (so θ corresponds to a third or fourth quadrant direction).

* The second of the identities in Equations 2 then implies that

$$cot^2(\theta) = csc^2(\theta) - 1 = (-3)^2 - 1 = 8 \Rightarrow cot(\theta) = \pm\sqrt{8} ,$$

  with the positive square root corresponding to the third quadrant possibility and the negative square root to the fourth quadrant possibility.

* In this example, if we add the information about whether we mean the third or fourth quadrant direction corresponding to $csc(\theta) = -3$, we can then also solve for the

cotangent, including the correct choice of sign, and, from this, use $sin(\theta) = 1/csc(\theta)$ and $cos(\theta) = cot(\theta)\,sin(\theta) = cot(\theta)/csc(\theta)$ to immediately determine the sine and cosine of the angle in question, and hence also the values of the other two trig functions, $sec(\theta) = 1/cos(\theta)$ and $tan(\theta) = sin(\theta)/cos(\theta)$.

* *Once more, the value of one of the two functions in the identity, together with the information about which of the two possible quadrants is the one we want to consider, allows us to determine all six trig functions for the direction in question.*

– Example 6:

* Suppose we thought we had an angle for which $sin(\theta) = 3/2$, forgetting that $3/2$ is not a possible value for the sine function. If we tried to used Equation 1 to solve for the cosine of this angle, we would find

$$cos^2(\theta) = 1 - sin^2(\theta) = 1 - \left(\frac{3}{2}\right)^2 = -\left(\frac{5}{4}\right) \ .$$

* The left hand side of this equation is, however, a square, hence ≥ 0, while the right hand side is < 0.

* It follows that there is no θ for which this equation is satisfied.

* Again, the equation "knows" about the fact that $3/2$ is not a possible value for the sine of any angle.

- Another piece of information embedded in the identities derived above is information we have already seen earlier about the allowed values of the sine and cosine and the secant and cosecant:
  - In the equation $sin^2(\theta) + cos^2(\theta) = 1$,
    * The left hand side is made up of a sum of two squares, each of which has to be ≥ 0.
    * Thus, neither term can, by itself, be > 1.
    * But if $sin^2(\theta)$ and $cos^2(\theta)$ both have to be ≤ 1 it follows that $sin(\theta)$ and $cos(\theta)$ both have to lie between -1 and 1, a fact we already found out earlier by other means.
    * The identity thus "knows" about the geometrical restrictions on the possible values of the sine and cosine functions.
  - Similarly, in the equation $tan^2(\theta) + 1 = sec^2(\theta)$,
    * The first term is a square and hence ≥ 0.
    * This implies that $sec^2(\theta)$ is ≥ 1 and hence that either $sec(\theta) \leq -1$ or $sec(\theta) \geq 1$, another fact we already found out early by other means.
    * Again the identity "knows" about the geometrical restrictions on the value of the secant function.
  - An argument similar to that just given for the secant shows that the equation $1 + cot^2(\theta) = csc^2(\theta)$ also requires that either $csc(\theta) \leq -1$ or $csc(\theta) \geq 1$, i.e., that the identity once more "knows" about the geometrical restrictions on the value of the cosecant function.

- One final piece of information contained the identities of Equations 2 is the following
  - First note that the square of any real number, positive, negative, or zero, is the same as the square of its "size" (absolute value).
  - The equation $tan^2(\theta) + 1 = sec^2(\theta)$ thus implies that, whenver the two functions are defined, the squared size of the secant is always 1 greater than the squared size of the tangent, and hence that the size (absolute value) of the secant is ALWAYS greater than the size (absolute value) of the tangent.
  - A similar argument, based on $1 + cot^2(\theta) = csc^2(\theta)$, shows that, whenever the cotangent and cosecant are defined, the size (absolute value) of the cosecant is ALWAYS greater than the size (absolute value) of the cotangent.

EXERCISES:

1. An angle θ is such that $sin(\theta) = 3/5$. In which two quadrants is it possible for the direction to which θ corresponds to lie? What is the value of $cos(\theta)$ in each of those two quadrants?

2. An angle θ is such that $cos(\theta) = -1/2$. In which two quadrants is it possible for the direction to which θ corresponds to lie? What is the value of $sin(\theta)$ in each of those two quadrants?

3. An angle θ is such that $cot(\theta) = -\sqrt{8}$. In which two quadrants is it possible for the direction to which θ corresponds to lie? What is the value of $csc(\theta)$ in each of those two quadrants?

4. An angle θ is such that $csc(\theta) = -\sqrt{10}$. In which two quadrants is it possible for the direction to which θ corresponds to lie? What is the value of $cot(\theta)$ in each of those two quadrants?

5. An angle θ is such that $sec(\theta) = -\sqrt{26}$. In which two quadrants is it possible for the direction to which θ corresponds to lie? What is the value of $tan(\theta)$ in each of those two quadrants?

6. An angle θ is such that $tan(\theta) = 3/4$. In which two quadrants is it possible for the direction to which θ corresponds to lie? What is the value of $sec(\theta)$ in each of those two quadrants?

7. An angle θ is such that $cos(\theta) = -4/5$. In which two quadrants is it possible for the direction to which θ corresponds to lie? What is the value of $cot(\theta)$ in each of those two quadrants? (Note, the last part asks for the values of the cotangent, NOT the sine.)

8. An angle θ is such that $sec(\theta) = 13/5$. In which two quadrants is it possible for the direction to which θ corresponds to lie? What is the value of $cot(\theta)$ in each of those two quadrants? (Note, the last part asks for the values of the cotangent, NOT the tangent.)

9. An angle θ is such that $sin(\theta) = -5/13$. In which two quadrants is it possible for the direction to which θ corresponds to lie? What is the value of all of the other five trig functions in each of those two quadrants?

10. An angle θ is such that $sec(\theta) = -\sqrt{17}$. In which two quadrants is it possible for the direction to which θ corresponds to lie? What is the value of all of the other five trig functions in each of those two quadrants?

11. An angle θ is such that $cot(\theta) = \sqrt{15}$. In which two quadrants is it possible for the direction to which θ corresponds to lie? What is the value of all of the other five trig functions in each of those two quadrants?