A FEW COMMENTS ABOUT IDENTITIES DESIGNED TO ALLOW "NEW" INFORMATION TO BE GOT FROM "OLD" INFORMATION (IDEALLY WITH MUCH LESS WORK THAN REQUIRED TO GET THE ORIGINAL "OLD" INFORMATION)

A common strategy in mathematics is to try to use already established ("old") information to simplify the task of obtaining new, but similar information. In the two sections of Chapter 5 which follow this one (Sections (d) and (e)), we will introduce at a number of identities which are of this type.

In order to understand the point of such identities, and hence be able to (i) remember them (or be able to reconstruct them) effectively and (ii) make effective use of them, it is important to approach them with this understanding of what they are meant to do.

In this section, we provide a few specific examples, accompanied by detailed comments which focus on this point, in order to prepare you for the next two sections.

The sample identities used below to illustrate the idea of the "new information from old" type identity are, in fact, all proved explicitly in the next two sections of this Chapter, so you don't have to worry if they don't seem familiar to you.

Seeing where they came from is not the point of this section; the point is, rather, to see how such identities are to be interpreted, and how being aware of how they are to be interpreted helps to make MUCH clearer how it is that they are to be used.

That said, some of the sample identities used below are based on geometrical ideas we have already met in Sections (e) and (f) of Chapter 3 of the course. (In those sections, we discussed the relations between the trig functions of first quadrant angles and those of the related angles in the second, third and fourth quadrants associated with right-angle triangles obtained by flopping the relevant first quadrant right-angle triangle over into the second, third and fourth quadrants.) You can review the discussion of these geometrical ideas, if you like, before proceeding to the rest of material below, but it is not actually necessary to do so to follow the main point we want to make here.

Full derivations, including simple geometrical explanations, of all of the sample identities discussed below, as well as of a number of additional identities of the "new information from old" type, will be given in the next two sections of this chapter.

A SUMMARY OF WHAT IS GOING TO BE DONE IN THE EXAMPLES OF "NEW INFORMATION FOR OLD" IDENTITIES WE WILL WORK THROUGH BELOW

- In all of the examples which follow, we'll imagine that, by some means, we've managed to do some work and figure out the values of all the trig functions of some angle (or set of angles).

- The angle(s) in question, together with the corresponding values for the six trig functions of each such angle, then represent our "old" information.

- We will then look at various identities, all of which involve expressing one of the trig functions of a "new" angle, related in some way to the "old" angle (or angles), solely in terms of the trig functions of the "old" angle(s).

- To help you in getting used to the point of such identities we will always put the trig function of the "new" angle (representing the new information we would ideally like to work out, with as little effort as possible) on the left hand side of the identity and the expression involving the trig function(s) of the "old" angle(s) (the information we already have, and, ideally, would like to use to simplify working out the desired new information) on the right hand side of the identity.

- The idea is thus that you should interpret the right hand side of each identity of this type as the "input information" and the left hand side as the "output information".

- Examples of what I've called "old" information above would be the already-worked-out values of the trig functions of the special first quadrant angles $\pi/6$, $\pi/4$ or $\pi/3$. (For each of these angles we already (in Section (d) of Chapter 2) went through the process of constructing a sample triangle having the angle in question as one of its non-right angles, figuring out the lengths of its sides using elementary geometry, and using this to work out the values of all six trig functions of that angle.)

- Another example of what we might call "old" information, and which we will use in some of the examples below, is the pair of results

$$cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}, \qquad and \qquad sin\left(\frac{2\pi}{5}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}\ .$$

  - The second of these equations in fact follows from the first if one uses the fact that $2\pi/5$ is a first quadrant angle (and hence has a positive sine) in combination with the Pythagoras' Theorem identity $sin^2(2\pi/5) + cos^2(2\pi/5) = 1$.

  - Having the sine and cosine values, we of course also have the values of the other four trig functions of $2\pi/5$ (which we have not bothered to write down explicitly here).

  - The cosine result is not something whose derivation I can show you here since it requires using some advanced information about complex numbers and the relation between the sine, cosine and exponential function, e^z, of a complex variable z.

- For the purposes of this section, however, it doesn't matter if you have no idea what any of what I just said means — the point is that, after doing a lot of work, someone in the third year complex variables calculus course can work out this information and provide us with the values of all six trig functions of $2\pi/5$, which then serves as potential "old" information for use as input to various "new information from old" identities.

- The key point, which will be illustrated in some of the examples below, is that "new information from old" identities allow us to very easily work out the values of the trig functions of new angles related in various ways to the "old" angle $2\pi/5$ WITHOUT HAVING TO DO ANY NEW COMPLICATED COMPLEX NUMBER ARITHMETIC AT ALL. This is why we like identities of this type.

• Now to the examples, which should help to make what has just been said a little less abstract.

• It is recommended that you return to the outline above for a second reading after you've had a look at the examples.

- Example 1: Two of the identities which will be proved in the next section of this chapter of the course (and which relate the values of the trig functions of the two non-right angles appearing in the same right-angle triangle) are

$$cos\left(\frac{\pi}{2} - \theta\right) = sin(\theta) \qquad and \qquad sin\left(\frac{\pi}{2} - \theta\right) = cos(\theta)\ .$$

  - If we take $\theta = \pi/6$ as our "old" angle on the right hand side of these identities, the related "new" angle appearing on the left hand sides is $\frac{\pi}{2} - \theta = \frac{\pi}{3}$, which is also one of the angles for which we already know the trig function values. Thus, in this case, although the identities are correct, they give us no new information.

  - In contrast, if we take $\theta = 2\pi/5$ as our "old" angle on the right hand sides, the related "new" angle appearing on the left hand sides is $\frac{\pi}{2} - \theta = \frac{\pi}{2} - \frac{2\pi}{5} = \frac{\pi}{10}$, which is an angle whose trig function values we have NOT met before.

  - Thus, we can use the identities above to incredibly easily work out the totally new results

$$cos\left(\frac{\pi}{10}\right) = sin\left(\frac{2\pi}{5}\right) = \sqrt{\frac{5+\sqrt{5}}{8}} \qquad and \qquad sin\left(\frac{\pi}{10}\right) = cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}$$

- Example 2: Two of the identities which will be proved in the next section of this chapter of the course (and which relate the values of the trig functions of a first quadrant angle to those of the second quadrant angle related to it by the "flopping-over" construction discussed in Sections (e) and (f) of Chapter 3) are

$$sin\,(\pi - \theta) \;=\; sin(\theta) \qquad and \qquad cos\,(\pi - \theta) \;=\; -\,cos(\theta) \;\;.$$

  - If we take $\theta = \pi/6$ as our "old" angle on the right hand side of these identities, the related "new" angle appearing on the left hand sides is $\pi - \theta = \frac{5\pi}{6}$. We, in fact, already used the geometry which underlies these identities to work out all six of the trig functions of $5\pi/6$ on page 6 of Section (f) of Chapter 3 of the course. Here, we repeat only the following two results, which follow from the identities above,

$$sin(5\pi/6) \;=\; sin(\pi/6) \;=\; 1/2, \qquad cos(5\pi/6) \;=\; -cos(\pi/6) \;=\; -\frac{\sqrt{3}}{2} \;\;.$$

  - If we take, instead, $\theta = 2\pi/5$ as our "old" angle on the right hand sides, the related "new" angle appearing on the left hand sides is $\pi - \theta = \pi - \frac{2\pi}{5} = \frac{3\pi}{5}$, which is an angle whose trig function values we have NOT met before.

  - Thus, we can use the identities to incredibly easily work out the totally new results

$$sin\left(\frac{3\pi}{5}\right) \;=\; sin\left(\frac{2\pi}{5}\right) \;=\; \sqrt{\frac{5+\sqrt{5}}{8}} \qquad and \qquad cos\left(\frac{3\pi}{5}\right) \;=\; -cos\left(\frac{2\pi}{5}\right) \;=\; -\left(\frac{\sqrt{5}-1}{4}\right) \;\;.$$

- Example 3: Four very important and useful identities, which will be proved in Section (e) of this chapter, are the sine and cosine addition formulas,

$$\begin{aligned} sin(\theta_1+\theta_2) &= sin(\theta_1)cos(\theta_2) + cos(\theta_1)sin(\theta_2) \\ sin(\theta_1-\theta_2) &= sin(\theta_1)cos(\theta_2) - cos(\theta_1)sin(\theta_2) \\ cos(\theta_1+\theta_2) &= cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2) \\ cos(\theta_1-\theta_2) &= cos(\theta_1)cos(\theta_2) + sin(\theta_1)sin(\theta_2) \ . \end{aligned}$$

  - For these identities, there are two "old" angles, θ_1 and θ_2, on the right hand (old information) side of the equations, while the "new" angle on the left hand (new information) side either $\theta_1+\theta_2$ or $\theta_1-\theta_2$.
  - For example, with $\theta_1=\pi/3$ and $\theta_2=\pi/4$, the new angles are $\theta_1+\theta_2=7\pi/12$ and $\theta_1-\theta_2=\pi/12$. For neither of these new angles do we currently know the values of any of the trig functions, so the addition formulas allow us to work out the sines and cosines of both of the new angles $7\pi/12$ and $\pi/12$, with each calculation taking only a single line of easily performed arithmetic. Once we have the sines and cosines, we can also work out any of the othr four trig functions of these angles as well.
  - We illustrate these calculations for the difference, $\pi/12=(\pi/3)-(\pi/4)$ (the analogous calculations for the sum, $7\pi/12=(\pi/3)+(\pi/4)$, will be performed explicitly in Section (e) of this chapter):

$$\begin{aligned} sin(\pi/12) &= sin(\pi/3)cos(\pi/4) - cos(\pi/3)sin(\pi/4) = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}} \\ cos(\pi/12) &= cos(\pi/3)cos(\pi/4) + sin(\pi/3)sin(\pi/4) = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}} \ . \end{aligned}$$

– We could, similarly, use these identities, together with our old information on the values of the trig functions for $\theta_1 = 2\pi/5$ and either $\theta_2 = \pi/6$, $\theta_2 = \pi/4$ or $\theta_2 = \pi/3$, to work out exact values for the sines and cosines (and hence also the values of any of the other four trig functions) of the angles

$$\frac{2\pi}{5}+\frac{\pi}{6}=\frac{17\pi}{30}, \quad \frac{2\pi}{5}-\frac{\pi}{6}=\frac{7\pi}{30}, \quad \frac{2\pi}{5}+\frac{\pi}{4}=\frac{13\pi}{20}, \quad \frac{2\pi}{5}-\frac{\pi}{4}=\frac{3\pi}{20}, \quad \frac{2\pi}{5}+\frac{\pi}{3}=\frac{11\pi}{15} \; and \; \frac{2\pi}{5}-\frac{\pi}{3}=\frac{\pi}{15},$$

six new angles whose trig function values we have not previously worked out. E.g.,

$$sin\left(\frac{3\pi}{20}\right) = sin\left(\frac{2\pi}{5}\right) cos\left(\frac{\pi}{4}\right) - cos\left(\frac{2\pi}{5}\right) sin\left(\frac{\pi}{4}\right) = \left(\sqrt{\frac{5+\sqrt{5}}{8}}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$cos\left(\frac{3\pi}{20}\right) = cos\left(\frac{2\pi}{5}\right) cos\left(\frac{\pi}{4}\right) + sin\left(\frac{2\pi}{5}\right) sin\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\sqrt{\frac{5+\sqrt{5}}{8}}\right)\left(\frac{1}{\sqrt{2}}\right) .$$

- Example 4: As we will show in Section (e) of this chapter, the following identities, called the "half-angle formulas" follow in a few lines as special cases of the cosine addition formula:

$$cos^2\left(\frac{\theta}{2}\right) = \frac{1+cos(\theta)}{2}, \qquad and \qquad sin^2\left(\frac{\theta}{2}\right) = \frac{1-cos(\theta)}{2}$$

– Once we know what quadrant the "old" angle θ, appearing on the right hand (old information) sides of these equations, lies in, and hence what quadrant the "new" angle $\theta/2$, appearing on the left hand (new information) sides, lies in, we can decide on whether to take the positive or negative square roots to get the correct values of $sin(\theta/2)$ and $cos(\theta/2)$.

- If we take the old angle on the right hand sides to be $\theta = \pi/3$, the new angle on the left hand sides is the already familiar angle $\theta/2 = \pi/6$ and so the identities, though true, do not provide us with any actually new information.

- However, if we take the old angle on the right hand sides to be $\theta = \pi/4$, the new angle on the left hand sides is $\theta/2 = \pi/8$, an angle we do not currently know the values of the trig functions for.

  * In this case, we know that the new angle $\theta/2 = \pi/8$ is a first quadrant angle, and hence has positive values for both its sine and cosine.

  * Thus the first of the half-angle identities tells us that

$$cos\left(\frac{\pi}{8}\right) = +\sqrt{\frac{1+cos(\pi/4)}{2}} = \sqrt{\frac{1+(1/\sqrt{2})}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

    while the second of the half-angle identities tells us that

$$sin\left(\frac{\pi}{8}\right) = +\sqrt{\frac{1-cos(\pi/4)}{2}} = \sqrt{\frac{1-(1/\sqrt{2})}{2}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

- Similarly, the recently obtained "old" results for $\theta = \pi/12$ given in Example 3 above, would, when used as input on the right hand (old information) side of the half-angle identities, immediately yield exact values for the sine and cosine (and hence also the other four trig functions) of the new angle $\theta/2 = \pi/24$. Using these results as a new set of old information on the right hand side of the half-angle formulas would then yield, in a similar way, exact values of the trig functions of the new half-angle $\pi/48$ and so on and so on.

- Note that, having the values for, say $sin(\pi/48)$ and $cos(\pi/48)$, we could use the sine and cosine addition formulas with $\theta_1 = \theta_2 = \pi/48$ to get the sine and cosine of $\theta_1 + \theta_2 = 2\pi/48 = \pi/24$, after which, using the addition formulas again with the new choice of input angles, $\theta_1 = \pi/24$ and $\theta_2 = \pi/48$, would allow us to get the sine and cosine of $\theta_1 + \theta_2 = 3\pi/48$, after which, using the addition formulas again with the new choice of input angles, $\theta_1 = 3\pi/48$ and $\theta_2 = \pi/48$, would allow us to get the sine and cosine of $\theta_1 + \theta_2 = 4\pi/48$, after which, by successively adding $\pi/48$ to the most recent output angle from the previous step, and using the addition formula yet again, we could eventually get exact values for the sines and cosines (and hence also the other four trig functions) of any integer multiple of $\pi/48$.

- It we instead had used the half-angle formula to start with input from the "old" results for $\theta = \pi/48$ and work out the output sine and cosine values for the new angle $\theta/2 = \pi/96$, we could have, similarly, used the addition formulas over and over again to get exact values for the trig functions of angles spaced $\pi/96$ radians apart around the plane!

- The power of the various identities used in the examples above is the way they allow us to so easily get new information in terms of old: once we have worked out the "old" input, however difficult it may have been to get it, there is suddenly a whole bunch of analogous information on the values of the trig functions of various new angles which we can get with only a rather small amount of arithmetic.

You might now want to return the summary just above the first of the examples above and re-read it before moving on to the last two sections of this Chapter (and the course).

EXERCISES:

(In the exercises that follow, all angles are given in radians.)

1. Noting that $\frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$, use the addition formulas to work out the exact expressions for all six trig functions of the angle $5\pi/12$.

2. Use the half angle formulas to work out the exact values of the sine and cosine of $\pi/5$.

3. Use the results obtained for $sin(\pi/12)$ and $cos(\pi/12)$ in Example 3 to work out the exact results for the sine and cosine of the new angle $11\pi/12$.

4. Use the results obtained in Example 4 for $sin(\pi/8)$ and $cos(\pi/8)$ to work out the exact values for the sine and cosine of the new angle $3\pi/8$.