TRIG IDENTITIES FOR ANGLES IN THE SAME RIGHT ANGLE TRIANGLE, AND FOR ANGLES RELATED BY THE "FLOPPING-OVER" CONSTRUCTION

In this section of the chapter, we show how to use some elementary geometric observations, combined with the basic geometric definitions of the various trig functions, to work out four groups of identities.

In order to understand the idea behind these identities, how it is they are to be interpreted and used, and why it is they might be useful, it is recommended that you read through the previous section, Section (c), of this chapter before starting on this section. Past experience shows that, for many students, the preliminary discussion contained in Section (c) makes understanding the material in the current section much easier than it would otherwise be.

Note that in the figures used in explaining the origin of these identities below, we have, to be specific, and in order to know how to draw the figures, taken the "old" angle θ to be a first quadrant angle. The identities that result from the arguments based on these figures are, however, valid for any angle θ. To illustrate how one goes about generalizing the arguments to see that they do, in fact, apply for any θ, we will explain the generalization in one of the four cases, though we will not give the details of the analogous generalizing arguments for the other three cases.

The identities we will discuss fall into the following four groups:

- Identities showing that each of the trig functions of the "new" angle $\frac{\pi}{2} - \theta$ is equal to a different one of the trig functions of the corresponding "old" angle θ (when θ is a first quadrant angle, $\frac{\pi}{2} - \theta$ is the other non-right angle in the right-angle triangle having θ as one of its non-right angles).

- Identities showing that each of the trig functions of the "new" angle $\pi - \theta$ is equal to either plus of minus the same trig function of the "old" angle θ (when θ is a first quadrant angle, $\pi - \theta$ is the second quadrant angle related to θ by the flopping-over construction discussed in Sections (e) and (f) of Chapter 3).

- Identities showing that each of the trig functions of the "new" angle $-\theta$ is equal to either plus or minus the same trig function of the "old" angle θ (when θ is a first quadrant angle, $-\theta$ is the fourth quadrant angle related to θ by the flopping-over construction discussed in Sections (e) and (f) of Chapter 3).

- Identities showing that each of the trig functions of the "new" angle $\pi + \theta$ is equal to either plus or minus the same trig function of the "old" angle θ (when θ is a first quadrant angle, $\pi + \theta$ is the third quadrant angle related to θ by the double-flopping construction discussed in Sections (e) and (f) of Chapter 3).

The main pieces of background needed to follow the material in this section are

- *the fact that the sum of the interior angles of any triangle is* π *radians and hence that, for a right angle triangle having one non-right angle* θ*, the other non-right angle is necessarily equal to* $(\pi/2) - \theta$*;*

- *the basic geometrical definitions of all six trig functions as the ratios of sides of right angle triangles, phrased in terms of the concepts "hypoteneuse", "adjacent" and "opposite" (discussed in Section (c) of Chapter 2 of the course);*

- *the algebraic relations* $sec(\theta) = 1/cos(\theta)$, $csc(\theta) = 1/sin(\theta)$, $tan(\theta) = sin(\theta)/cos(\theta)$ *and* $cot(\theta) = cos(\theta)/sin(\theta)$ *which allow the tangent, cotangent, secant and cosecant functions to be written as ratios involving the sine and cosine;*

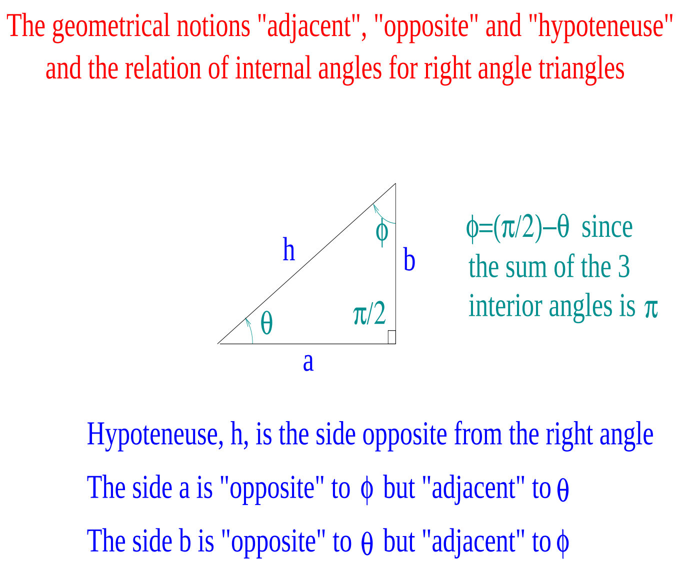
- *the unit circle picture for generalizing the sine and cosine functions, and from these, also the tangent, cotangent, secant and cosecant functions, from Quadrant I to Quadrant II, III and IV values of the relevant angle (discussed in Sections (b) and (c) of Chapter 3 of the course); and*

- *the alternate "flopped-over" triangle perspective for thinking about the relation between trig functions of first quadrant angles and the appropriately related second, third and four quadrant angles (discussed in Sections (e) and (f) of Chapter 3 of the course).*

OTHER-ANGLE-IN-THE-SAME-RIGHT-ANGLE TRIANGLE IDENTITIES

- The relation of the two non-right angles in the same right-angle triangle, and the meaning of the concepts "opposite" and "adjacent", were illustrated earlier in Section (c) of Chapter 2 of the course. This information is repeated here, in Figure 1, for your convenience.

FIG. 1:



- In the present context, θ is the "old" angle and $\phi = \frac{\pi}{2} - \theta$ is the "new" angle.

- We want to figure out how to obtain the "new" results, $sin(\phi)$, $cos(\phi)$, $tan(\phi)$, $sec(\phi)$, $csc(\phi)$ and $cot(\phi)$ in terms of the set of "old" (imagined already known) values, $sin(\theta)$, $cos(\theta)$, $tan(\theta)$, $sec(\theta)$, $csc(\theta)$ and $cot(\theta)$.

- This is incredibly easy to do since the side adjacent to ϕ is opposite to θ and the side opposite to ϕ is adjacent to θ.

- Denoting the sides adjacent and opposite to θ as $adjacent_\theta$ and $opposite_\theta$, and the sides adjacent and opposite to ϕ as $adjacent_\phi$ and $opposite_\phi$, this means $opposite_\phi = adjacent_\theta$ and $adjacent_\phi = opposite_\theta$.

- Thus, for example,

$$
\begin{aligned}
sin(\phi) = sin\left(\frac{\pi}{2} - \theta\right) &= opposite_\phi/hypoteneuse = adjacent_\theta/hypoteneuse = cos(\theta) \\
cos(\phi) = cos\left(\frac{\pi}{2} - \theta\right) &= adjacent_\phi/hypoteneuse = opposite_\theta/hypoteneuse = sin(\theta) \\
tan(\phi) = tan\left(\frac{\pi}{2} - \theta\right) &= opposite_\phi/adjacent_\phi = adjacent_\theta/opposite_\theta = cot(\theta)
\end{aligned}
$$

each of these results expressing the "new" information on the left hand side in terms of "old" information on the right hand side. You should be able to easily derive for yourself the following analogous "new"-in-terms-of-"old" identities for the remaining three trig functions of $(\pi/2) - \theta$ (try this to make sure you understand how easy it is)

$$
sec\left(\frac{\pi}{2} - \theta\right) = csc(\theta), \quad csc\left(\frac{\pi}{2} - \theta\right) = sec(\theta), \quad and \quad cot\left(\frac{\pi}{2} - \theta\right) = tan(\theta) \ .
$$

"FLOPPED-OVER-TRIANGLE" IDENTITIES

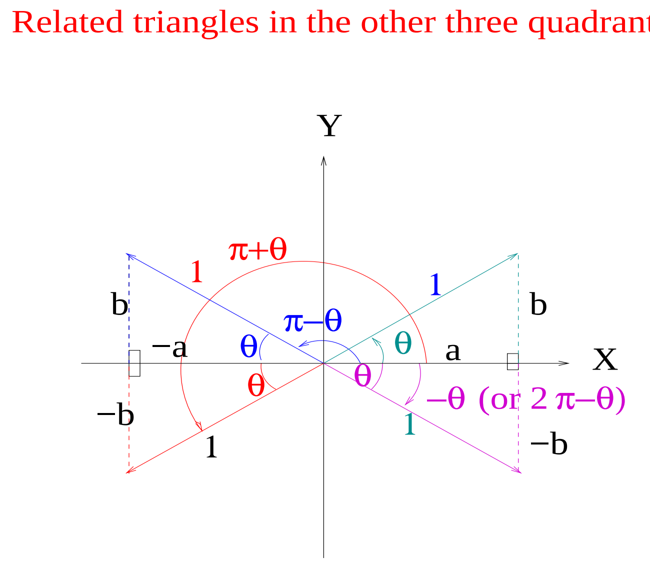
- When we discussed earlier, in Section (f) of Chapter 3 of the course, how to get the values of the trig functions corresponding to directions one-third, one-half, and two-thirds of the way through the second, third and fourth quadrants in terms of the trig functions for the special first quadrant directions $\pi/6$, $\pi/4$ and $\pi/3$, we imagined taking the relevant first quadrant angle triangle and flopping it over into each of the other three possible quadrants.

- Flopping from the right to the left side of the y-axis changed the sign of the x coordinate of the tip of the hypoteneuse without changing the y coordinate, while flopping from above to below the x-axis changed the sign of the y coordinate without changing the x coordinate.

- In that discussion, we considered the original first quadrant triangle and related second, third and fourth quadrant triangles shown in Figure 2 below, imagining that the angle θ was one of our special first quadrant angles $\pi/6$, $\pi/4$ or $\pi/3$. The geometry of the Figure, however, remains valid for any first quadrant angle θ.

- The figure shows all three "new" angles, $\pi - \theta$ (second quadrant), $\pi + \theta$ (third quadrant) and $-\theta$ (or, equivalently, $2\pi - \theta$) (fourth quadrant) obtained by the various flopping-over operations.

FIG. 2:



- As shown in the figure, if the side adjacent to θ in the original first quadrant triangle is a and that opposite to θ is b, then flopping over from the right to the left changes the "old" horizontal (adjacent) side a into the "new" horizontal (adjacent) side $-a$, while flopping over from above to below the x-axis changes the "old" vertical (opposite) side b into the "new" vertical (opposite) side $-b$. (All sides of all four triangles in the figure have been labelled accordingly.)

- It is now a straightforward matter to work out the trig functions of the "new" angles in terms of the corresponding trig functions of the original angle θ.
  - For the second quadrant case, with "new" angle $\pi - \theta$: The facts that the "new" opposite and hypoteneuse are the same as the "old" opposite and hypoteneuse, respectively, while the "new" adjacent is the negative of the "old" adjacent allow us to immediately see that

$$\begin{aligned} sin(\pi - \theta) &= sin(\theta) \\ cos(\pi - \theta) &= -cos(\theta) \\ tan(\pi - \theta) &= -tan(\theta) \\ sec(\pi - \theta) &= -sec(\theta) \\ csc(\pi - \theta) &= csc(\theta) \\ cot(\pi - \theta) &= -cot(\theta) \end{aligned} \quad (1)$$

    where in all cases the identities give the desired "new" value on the left hand side in terms of the "old" (assumed already known) value on the right hand side.

– For the third quadrant case, with new angle $\pi + \theta$: Both the horizontal (adjacent) and vertical (opposite) sides have undergone a sign change. This allows us to immediately determine "new"-in-terms-of-"old" identities between the trig functions of the new third quadrant angle $\pi + \theta$ and those of the old first quadrant angle θ. The results for the sine, cosine and tangent are given below, with the analogous results for the secant, cosecant, and cotangent left to the exercises at the end of this section:

$$
\begin{aligned}
sin(\pi + \theta) &= -sin(\theta) \\
cos(\pi + \theta) &= -cos(\theta) \\
tan(\pi + \theta) &= [-sin(\theta)]/[-cos(\theta)] = tan(\theta) \ .
\end{aligned}
\tag{2}
$$

– For the fourth quadrant case, with new angle $-\theta$: The analogous identities giving the trig functions of the "new" fourth quadrant related angle, $-\theta$ in terms of the corresponding trig function of the "old" first quadrant angle θ are also left as exercises. Results for the sine and cosine, often called the "symmetry properties of the sine and cosine", are

$$
\begin{aligned}
sin(-\theta) &= -sin(\theta) \\
cos(-\theta) &= cos(\theta) \ .
\end{aligned}
\tag{3}
$$

A SHORT FINAL COMMENT: As noted earlier, although the figures above have been drawn assuming θ is a first quadrant angle, all four sets of identities remain valid for any θ. We illustrate this point for the case of the identities between the trig functions of the new angle $\pi + \theta$ and those of the old angle θ. Arguments for the other three sets of identities can also easily (if tediously) be made but are not particularly intuitively illuminating, and so will not be gone through in detail here.

HOW TO GENERALIZE THE IDENTITIES INVOLVING $\pi + \theta$ AND θ TO ARBITRARY θ

- *Regardless of what value* θ *has,* $\pi + \theta$ *is* π *radians greater than* θ*, which implies that the* $\pi + \theta$ *direction lies exactly one half of a full revolution around clockwise from* θ *direction (i.e. the hypoteneuses for the right triangles associated with* θ *and* $\pi + \theta$ *point in exactly opposite directions).*

- *Thus (i) if* θ *is a second quadrant angle,* $\pi + \theta$ *is a fourth quadrant angle, (ii) if* θ *is a third quadrant angle,* $\pi + \theta$ *is a first quadrant angle and (iii) if* θ *is a fourth quadrant angle,* $\pi + \theta$ *is a second quadrant angle.*

- *If both hypoteneuses have length* 1*, the fact that they are half a revolution apart means that the* x *coordinates of the tips of the two hypoteneuses are the negatives of one another and, the* y *coordinates of the tips of the two hypoteneuses are, similarly, also the negatives of one another.*

- *Using the unit circle perspective, this imples that* $cos(\pi + \theta) = -cos(\theta)$ *and* $sin(\pi + \theta) = -sin(\theta)$*, the same results as we had when* θ *was a first quadrant angle, regardless of whether* θ *is a first, second, third, or fourth quadrant angle.*

- *Since the secant, cosecant, tangent and cotangent can be written as ratios involving the sine and cosine, the fact that sine and cosine identities in the case of general* θ *values are the same as we found for first quadrant* θ *values, it follows that the relations between the secants of* θ *and* $\pi + \theta$*, the cosecants of* θ *and* $\pi + \theta$*, the tangents of* θ *and* $\pi + \theta$*, and the cotangents of* θ *and* $\pi + \theta$*, also are the same as they were for first quadrant values of* θ*, as claimed.*

EXAMPLES

In the examples which follow we will take as given the values of the trig functions of the special first quadrant angles $\pi/6$, $\pi/4$ and $\pi/3$, as well as the following values, either given, or worked out in examples, in Section (c) of this chapter:

$$cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}, \qquad and \qquad sin\left(\frac{2\pi}{5}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}$$

$$sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}, \qquad and \qquad cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

- Example 1: Since we previously worked out the sine and cosine of $\pi/8$, the fact that $3\pi/8 = (\pi/2) - (\pi/8)$ allows us use the identities $sin\left(\frac{\pi}{2}-\theta\right) = cos(\theta)$ and $cos\left(\frac{\pi}{2}-\theta\right) = sin(\theta)$, to work out

$$sin\left(\frac{3\pi}{8}\right) = sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right) = cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$cos\left(\frac{3\pi}{8}\right) = cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right) = sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}\ .$$

The remainder of the trig functions of $3\pi/8$ follow from $sec(3\pi/8) = 1/cos(3\pi/8)$, $csc(3\pi/8) = 1/sin(3\pi/8)$, $tan(3\pi/8) = sin(3\pi/8)/cos(3\pi/8)$ and $cot(3\pi/8) = cos(3\pi/8)/sin(3\pi/8)$.

- Example 2: If we are asked to find the sine and cosine of $3\pi/4$, we can (i) notice that $3\pi/4$ is a second quadrant angle, equal to $\pi - (\pi/4)$ and (ii) remember that we know already $sin(\pi/4) = cos(\pi/4) = 1/\sqrt{2}$.

  The identities relating the trig functions of θ and $\pi - \theta$ can then be used to obtain the trig functions of $3\pi/4$. (Remember that the identities in question are based on the geometric observation that flopping the $\pi/4$ triangle over from the first to the second quadrant changes the sign of the horizontal side (which determines the cosine function) but leaves the vertical side (which determines the sine function) unchanged.)

  The results for the sine and cosine are therefore

$$sin\left(\frac{3\pi}{4}\right) = sin\left(\pi - \frac{\pi}{4}\right) = sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$
$$cos\left(\frac{3\pi}{4}\right) = cos\left(\pi - \frac{\pi}{4}\right) = -cos\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}} .$$

- Example 3: To find the tangent of $7\pi/6$, we can (i) notice that $7\pi/6 = \pi + (\pi/6)$ and (ii) remember that we know already $sin(\pi/6) = 1/2$ and $cos(\pi/6) = (\sqrt{3})/2$. The identities relating the trig functions of θ and $\pi + \theta$ (which follow from the underlying geometry that the double flop from the first to the third quadrant changes the sign of BOTH the sine and cosine, and hence leaves the sign of the ratio of the two (the tangent) unchanged) can thus be used to obtain

$$tan\left(\frac{7\pi}{6}\right) = tan\left(\pi + \frac{\pi}{6}\right) = +tan\left(\frac{\pi}{6}\right) = \frac{1/2}{(\sqrt{3})/2} = \frac{1}{\sqrt{3}} .$$

- Example 4: To find the trig functions of $7\pi/4$ we can (i) notice that $7\pi/4 = 2\pi - (\pi/4)$, which corresponds to the same physical direction as $-\pi/4$, and (ii) remember that we know already $sin(\pi/4) = cos(\pi/4) = 1/\sqrt{2}$. Since flopping the $\pi/4$ triangle over from the first to the fourth quadrant changes the sign of the vertical side (which determines the sine function) and leaves the horizontal side (which determines the cosine function) unchanged, it follows that

$$\begin{aligned} sin(7\pi/4) &= sin(-\pi/4) = -sin(\pi/4) = -1/\sqrt{2} \\ cos(7\pi/4) &= cos(-\pi/4) = cos(\pi/4) = 1/\sqrt{2} \\ sec(7\pi/4) &= sec(-\pi/4) = sec(\pi/4) = \sqrt{2} \\ tan(7\pi/4) &= tan(-\pi/4) = -tan(\pi/4) = -1, \ \ etc. \end{aligned}$$

- Example 5: Another example of the use of the identities which relate the trig functions of the two angles, θ and $(\pi/2) - \theta$, in the same right-angle triangle is the case of the trig functions of $\pi/10 = (\pi/2) - (2\pi/5)$, worked out in Section (c) of this chapter, and repeated here. We have already been given the values of the trig functions of $2\pi/5$ (above) and, though these require advanced techniques involving complex arithmetic to derive, once someone has derived them and given us their values, the identities allow us to obtain, with practically no effort at all, the results

$$\begin{aligned} sin\left(\frac{\pi}{10}\right) &= sin\left(\frac{\pi}{2} - \frac{2\pi}{5}\right) = cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4} \\ cos\left(\frac{\pi}{10}\right) &= cos\left(\frac{\pi}{2} - \frac{2\pi}{5}\right) = sin\left(\frac{2\pi}{5}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}. \end{aligned}$$

- Example 6: To find the trig functions of $9\pi/8$ we can (i) notice that $9\pi/8$ is a third quadrant angle, with $9\pi/8 = \pi + (\pi/8)$, and (ii) remember that we have previously worked out the trig functions of $\pi/8$. Since double-flopping the $\pi/8$ triangle from the first to the third quadrant changes the signs of both the vertical side (which determines the sine function) and the horizontal side (which determines the cosine function),

$$
\begin{aligned}
sin(9\pi/8) &= -sin(\pi/8) = -\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \\
cos(9\pi/8) &= -cos(\pi/8) = -\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} \\
csc(9\pi/8) &= \frac{1}{sin(9\pi/8)} = \frac{1}{-sin(\pi/8)} = -csc(\pi/8) = -\sqrt{\frac{2\sqrt{2}}{\sqrt{2}-1}} \\
cot(9\pi/8) &= \frac{cos(9\pi/8)}{sin(9\pi/8)} = \frac{-cos(\pi/8)}{-sin(\pi/8)} = cot(\pi/8) = \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}, \ etc.
\end{aligned}
$$

- Example 7: As a reminder of the fact that the identities are not restricted to first quadrant values of the "old" angle θ, we can work out the sine and cosine of the "new" angle $-3\pi/4$ from the results obtained for the now "old" angle $3\pi/4$, in Example 2 above:

$$
\begin{aligned}
sin\left(-\frac{3\pi}{4}\right) &= -sin\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}} \\
cos\left(-\frac{3\pi}{4}\right) &= cos\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}} \ .
\end{aligned}
$$

- Example 8: To illustrate that there may be more than one way to work out the values of the trig functions of a new angle, let's work out the sine and cosine of the "new" angle $-3\pi/4$ of Example 7 in a different way.

  Here, we note that $-3\pi/4$ is a third quadrant angle, corresponding to the same physical direction as $2\pi - (3\pi/4) = 5\pi/4$.

  Then, since $5\pi/4 = \pi + (\pi/4)$, we can use the identities relating the trig functions of θ and $\pi + \theta$ as an alternative to following the approach used in Example 7 (which instead used the identities relating the trig functions of θ and $-\theta$). The results obtained are, of course, the same. The explicit calculation is as follows:

$$sin\left(-\frac{3\pi}{4}\right) = sin\left(\frac{5\pi}{4}\right) = sin\left(\pi + \frac{\pi}{4}\right) = -sin\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$

$$cos\left(-\frac{3\pi}{4}\right) = cos\left(\frac{5\pi}{4}\right) = cos\left(\pi + \frac{\pi}{4}\right) = -cos\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}} .$$

- Further examples, which you should try for yourself, are given in the exercises below.

EXERCISES:

(In the exercises that follow, all angles are given in radians.)

1. Work out the identities which allow you to express $sec(\pi + \theta)$, $csc(\pi + \theta)$, and $cot(\pi + \theta)$ in terms of the trig functions of the angle θ.

2. Work out the identities which allow you to express $sec(-\theta)$, $csc(-\theta)$, $tan(-\theta)$ and $cot(-\theta)$ in terms of the trig functions of the angle θ.

In the remaining questions, figure out how express the given angle ϕ as one of $(\pi/2) - \theta$, $\pi - \theta$, $\pi + \theta$ or $-\theta$ (or, in the last case, equivalently, $2\pi - \theta$) for one of the angles θ for which we already know the values of the trig functions. Consider as "already known" all of the trig functions of the special first quadrant angles $\pi/6$, $\pi/4$ and $\pi/3$, as well as the results $sin(2\pi/5) = \sqrt{(5 + \sqrt{5})/8}$, $cos(2\pi/5) = (\sqrt{5} - 1)/4$, which were quoted already above.)

3. Determine the sine, cosine, secant and tangent of the angle $\phi = 7\pi/5$.

4. Determine the sine, cosine, cosecant and cotangent of the angle $\phi = 8\pi/5$.

5. Determine the sine, cosine, cosecant and cotangent of the angle $\phi = 5\pi/3$.

6. Determine the sine, cosine, secant and cotangent of the angle $\phi = 5\pi/4$.

7. Determine the sine, cosine, cosecant and tangent of the angle $\phi = -5\pi/4$.