# THE TRIG FUNCTIONS FOR NON-FIRST-QUADRANT DIRECTIONS

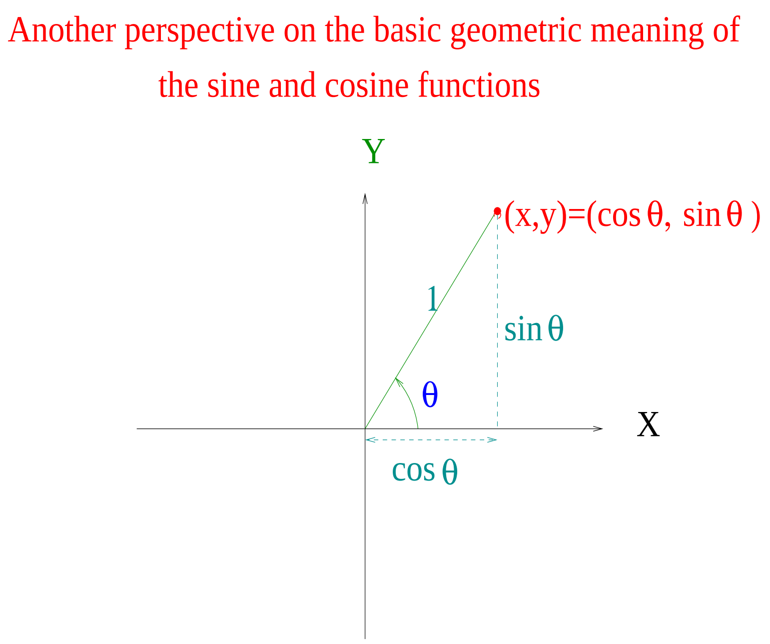
*The key pieces of background information for this section of the course are:*

- *the "unit circle picture" for the sine and cosine functions for first quadrant angles (which can be reviewed by reading Section (b) of this Chapter of the course) and*
- *the algebraic relations amongst the different trig functions of the same first quadrant angle, especially the relations which allow one to express the tangent, cosecant, secant and cotangent functions in terms of the sine and cosine (these relations are repeated below, and can be reviewed by reading (or re-reading) Section (a) of this Chapter).*

To this point in the course, our discussions have focussed on the case where the angle under consideration was one of the two non-right interior angles of a right-angle triangle. All such angles automatically lie between $0$ and $\pi/2$ radians, and hence correspond to first quadrant directions. In this section we discuss how to naturally (and geometrically) generalize the trig functions to cases where the angle in question corresponds to a non-first quadrant direction.

The easiest way to generalize the definitions of the trig functions to non-first-quadrant angles is to start from the unit circle picture for the sine and cosine of first quadrant angles. Figure 1, from Section (b) of this chapter, is repeated as a quick reminder at the top of the next page.

FIG. 1:




The generalization argument then goes as follows:

- In words, the unit circle picture shown in Figure 1 gives a representation of the sine and cosine in which $cos(\theta)$ is interpreted as the $x$-coordinate of the tip of a line of length $1$ extending out from the origin in the $\theta$ direction and $sin(\theta)$ is interpreted as the $y$-coordinate of the tip of this same line, i.e., in which the $(x, y)$ coordinates of the tip of the line of length $1$ extending out from the origin in the $\theta$ direction determine the values of the sine and cosine of the angle $\theta$ via $(x, y) = (cos(\theta), sin(\theta))$.

- While the picture still involves only first quadrant angles $\theta$, there is nothing to prevent us from now rotating the hypoteneuse line into a different directions which no longer lies in the first quadrant.

- This provides the following straightforward way of generalizing the geometrical definitions of $sin(\theta)$ and $cos(\theta)$ so that they apply to angles $\theta$ corresponding to *any* direction in the plane:

> DEFINITION: The $(x, y)$ coordinates of the tip of a line of length $1$, beginning at the origin and pointing in the $\theta$ direction, will in all cases be interpreted to be $(x, y) = (cos(\theta),\, sin(\theta))$

- The generalizations of the other four trig functions $tan(\theta)$, $cot(\theta)$, $sec(\theta)$ and $csc(\theta)$, to non-first-quadrant angles can then be defined by using this generalization of the sine and cosine, together with the algebraic relations

$$csc(\theta) = \frac{1}{sin(\theta)}, \qquad sec(\theta) = \frac{1}{cos(\theta)}$$
$$tan(\theta) = \frac{sin(\theta)}{cos(\theta)} \qquad cot(\theta) = \frac{cos(\theta)}{sin(\theta)}$$

which were already shown to be valid for first quadrant angles in Section (a) of this chapter.

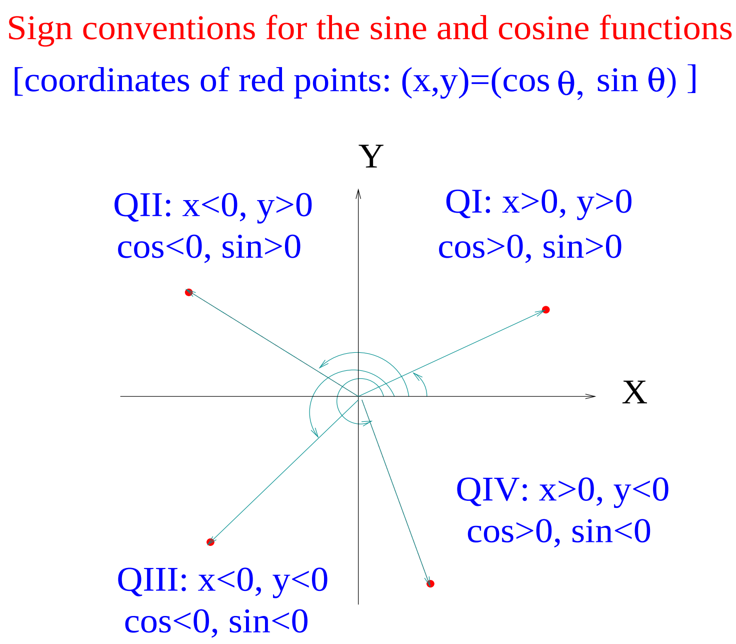
With these basic definitions for the generalizations of the six trig functions to non-first quadrant angles, some qualitative features of the various functions become immediately obvious:

- The signs of the sine and cosine in the various quadrants:
  - Since the $x$ coordinate of the tip of the line is $< 0$ if the line points to the left of the $y$-axis (in a second or third quadrant direction) but $> 0$ if the line points to the right of the $y$-axis (in a first or fourth quadrant direction), $cos(\theta)$ is $> 0$ for first and fourth quadrant angles, and $< 0$ for second and third quadrant angles.

– Similarly, since the $y$ coordinate of the tip of the line is $> 0$ if the tip of the line points above the $x$-axis (in a first or second quadrant direction), but $< 0$ if the tip of the line points below the $x$-axis (in a third or fourth quadrant direction), $sin(\theta)$ is $> 0$ for first and second quadrant angles, and $< 0$ for third and fourth quadrant angles.

– A summary of these observations about the signs of the sine and cosine function in the various quadrants in provided in Figure 2.

FIG. 2:



- The signs of the cosecant, secant, tangent and cotangent in the various quadrants:

  The results that $sin(\theta)$ is $> 0$ in Quadrant I and II and $< 0$ in Quadrants III and IV, while $cos(\theta)$ is $> 0$ in Quadrant I and IV and $< 0$ in Quadrants II and III, combined with the algebraic relations written down above, allow us to immediately find out about the signs of the other four trig functions in the various quadrants as well.

  – Example 1: The secant function

    * The basic algebraic relation for the secant is $sec(\theta) = 1/cos(\theta)$.
    * Thus, $sec(\theta)$ is $> 0$ whenever $cos(\theta)$ is $> 0$ (in Quadrants I and IV) and $< 0$ whenever $cos(\theta)$ is $< 0$ (in Quadrants II and III).

  – Example 2: The cotangent function

    * The basic algebraic relations for the cotangent is $cot(\theta) = cos(\theta)/sin(\theta)$
    * Thus in Quadrant I, where the sine and cosine are both $> 0$, $cot(\theta) > 0$.
    * Similarly, in Quadrant II, where $sin(\theta) > 0$ and $cos(\theta) < 0$, $cot(\theta) < 0$.
    * You should be able to use similar arguments to verify that $cot(\theta)$ is $> 0$ in Quadrant III and $< 0$ in Quadrant IV.

  – You are asked to determine for yourself the signs of the cosecant and tangent functions in the various quadrants in the exercises below.

- Directions for which the secant, cosecant, tangent or cotangent are undefined
  - Case 1: The cosecant and the cotangent functions
    * Recall that $sin(\theta)$ appears in the denominator of the expressions for both $csc(\theta)$ and $cot(\theta)$.
    * Since division by zero is undefined, this implies that $csc(\theta)$ and $cot(\theta)$ will be undefined for any $\theta$ for which $sin(\theta) = 0$.
    * This happens whenever the $y$ coordinate of the tip of the hypoteneuse of length *1* is *0* (which means that the tip of the hypoteneuse lies on either the *+x* or *-x* axis.
    * The angles for which this occurs are

$$+x \ \ direction: \ \ \theta = 0, \, , \, \pm 2\pi \, , \, \pm 4\pi \, , \, \cdots$$
$$-x \ \ direction: \ \ \theta = \pm \pi \, , \, \pm 3\pi \, , \, \pm 5\pi \, , \, \cdots$$

  - Case 2: The secant and the tangent functions
    * Recall that $cos(\theta)$ appears in the denominator of the expressions for both $sec(\theta)$ and $tan(\theta)$.
    * Thus, $sec(\theta)$ and $tan(\theta)$ are undefined for all $\theta$ for which $cos(\theta) = 0$.
    * It is left as one of the exercises below for you to figure out what the directions are in which this happens, and the angles to which such directions correspond.

- Note that, in contrast to the tangent, cotangent, secant and cosecant, each of which has two directions in which it is undefined, the sine and cosine are defined for ALL directions (since the tip of the hypoteneuse of length *1* has to be somewhere, and hence have well-defined $x$ and $y$ coordinates, regardless of what direction the hypoteneuse points in).

**EXERCISES:**

1. Determine the sign (positive or negative) of $csc(\theta)$ for $\theta$ in each of Quadrants II, III and IV.

2. Determine the sign (positive or negative) of $tan(\theta)$ for $\theta$ in each of Quadrants II, III and IV.

3. Determine those directions for which $cos(\theta) = 0$ and hence for which the functions $sec(\theta)$ and $tan(\theta)$ (in which $cos(\theta)$ appears in the denominator of the relevant algebraic expression) are undefined. Give three examples of angles which correspond to each such direction.