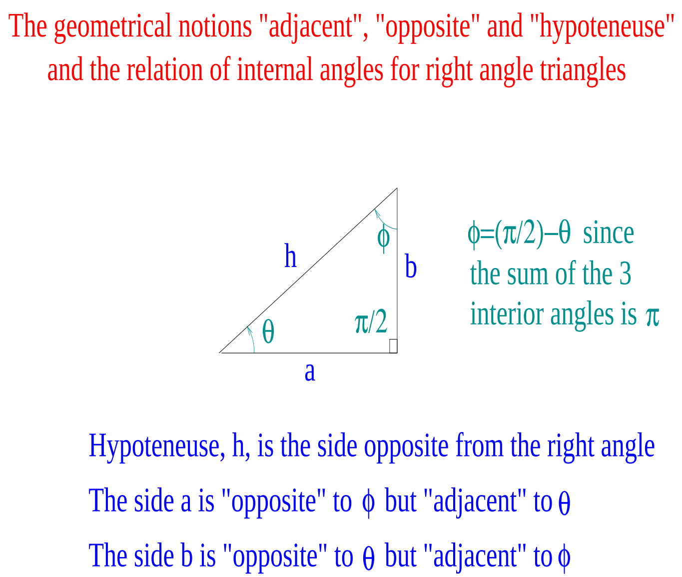
EXACT TRIG FUNCTIONS VALUES FOR THE ANGLES $\pi/6$, $\pi/4$ AND $\pi/3$

In this section we show how to use the basic geometric definitions to work out exact values for all six of the trig functions of three special first quadrant angles. These angles are special because, for each such angle, it is easy to figure out, using only elementary geometry, how to draw an example of a right-angle triangle having that angle as an interior angle with all side lengths exactly determined. As soon as this is done, the sides can be identified as being the "hypoteneuse", or being "adjacent" or "opposite" to the angle of interest, at which point the values of the trig functions for the angle in question can witten down using the basic geometric ratio definitions. Figure 1 is provided as a reminder of the geometrical meaning of the terms "hypoteneuse", "adjacent" and "opposite".

FIG. 1:



- *To follow the rest of the discussion in this section, you will need to remember the geometric definitions of the six trig functions in terms of the concepts "hypoteneuse", "adjacent" and "opposite".*

- *It is very important to get used to these definitions, since they represent the most basic meanings of the trig functions.*

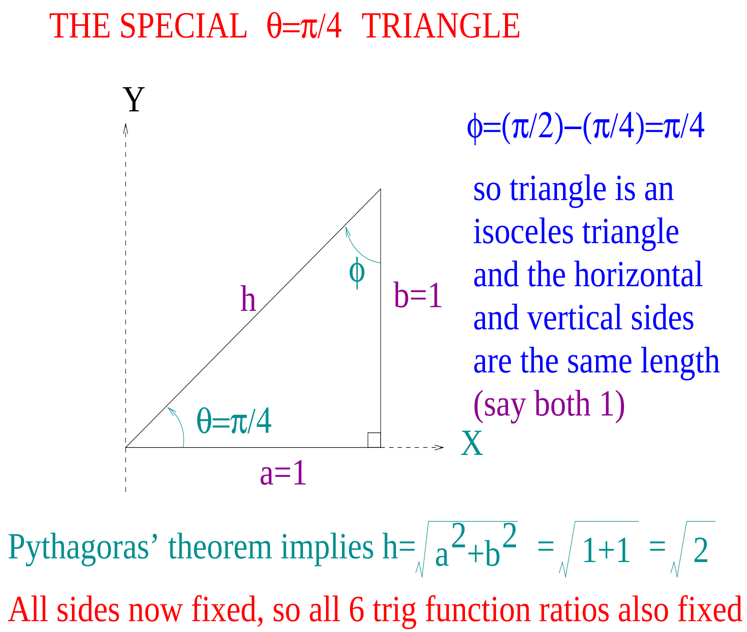
- *For students who are still in the process of becoming familiar with this information (which was introduced in the previous section of this Chapter) we repeat these definitions below.*

$$\begin{aligned} sin(\theta) &= opposite/hypoteneuse \\ cos(\theta) &= adjacent/hypoteneuse \\ tan(\theta) &= opposite/adjacent \\ csc(\theta) &= hypoteneuse/opposite \\ sec(\theta) &= hypoteneuse/adjacent \\ cot(\theta) &= adjacent/opposite \ . \end{aligned}$$

- *If you feel you are not yet familiar with these definitions, it is strongly recommended that you use them to work out some examples for specific sample right-angle triangles. Exercises of this type were provided at the end of the previous section of this Chapter, and some additional Exercises, related to the special angles discussed in this section, provided at the end of this section.*

THE $\theta = \pi/4$ CASE:

FIG. 2:



- An example of a right-angle triangle with one angle equal to $\pi/4$ is shown in Figure 2
- The other non-right angle in this triangle, ϕ, is thus equal to $\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$
- The triangle is thus isoceles, with opposite and adjacent sides having the same length

- We can use any sized example of a triangle with this shape to work out the trig functions of $\pi/4$; for simplicity we pick the one where the size of the two equal non-hypoteneuse sides is 1

- Pythagoras' Theorem then implies that the length of the hypoteneuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$

- With adjacent and opposite sides having length 1 and the hypoteneuse having length $\sqrt{2}$, the geometric ratio definitions of the trig functions allow us to get the exact values of all six trig functions of $\pi/4$. For example,

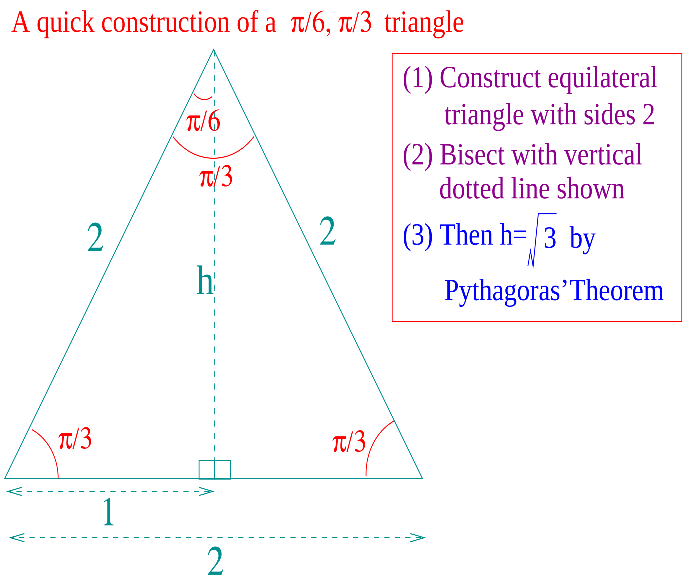
$$\begin{aligned} sin(\pi/4) &= opposite/hypoteneuse = 1/\sqrt{2} = (\sqrt{2})/2 \\ tan(\pi/4) &= opposite/adjacent = 1/1 = 1 \\ sec(\pi/4) &= hypoteneuse/adjacent = \sqrt{2}/1 = \sqrt{2} \end{aligned} \quad (1)$$

  (the other three trig functions for the angle $\pi/4$ are left for you to work out from the figure as part of the Exercises at the end of this section)

- Note that NONE of these values have to be memorized. It is trivial to reconstruct the example $\pi/4$ right angle triangle and then just read off the values using the geometric definitions of the various trig functions (of course you have to familiarize yourself with the basic geometrical definitions first, but this is something it is crucial for you to do anyway)

THE $\theta = \pi/6$ AND $\pi/3$ CASES:

FIG. 3:



- Figure 3 shows an equilateral triangle, with sides of length *2*. All three internal angles are equal, and hence have to each be equal to $\pi/3$. The figure shows how to use this triangle to construct a sample right-angle triangle having one internal non-right angle equal to $\pi/3$ radians. (An expanded version of the discussion is also given below.)

- With one of the two non-right angles in such a right-angle triangle being equal to $\frac{\pi}{3}$, the other is $\frac{\pi}{2} - \frac{\pi}{3} = \frac{3\pi}{6} - \frac{2\pi}{6} = \frac{\pi}{6}$, so the "special" angles $\pi/6$ and $\pi/3$ in fact lie in the same right-angle triangle

- In Figure 3, the original equilateral triangle has been split into two equal parts by the dotted internal line bisecting the upper $\pi/3$ angle.

- Since the angle bisected by the dotted line was $\pi/3$ to begin with, each half of the upper angle is $\pi/6$ and the two half triangles represent two identical copies of a right-angle triangle having interior non-right angles equal to $\pi/6$ and $\pi/3$ and hypoteneuse of length *2*.

- Since the dotted line is a bisector, the base of each half triangle (which lies adjacent to the $\pi/3$ angle) is of length $(1/2)\,2 = 1$.

- The length of the dotted vertical line, denoted h in the figure, is then obtained from Pythagoras' Theorem: $1^2 + h^2 = 2^2 = 4$, which implies $h^2 = 3$ or $h = \sqrt{3}$

- This side, of length $\sqrt{3}$, is opposite to the $\pi/3$ angle (adjacent to the $\pi/6$ angle) in each of the two "half triangles"

- One thus has an example of a triangle with non-right angles $\pi/6$ and $\pi/3$ for which
  - the hypoteneuse has length *2*,
  - the side opposite to $\pi/3$ (adjacent to $\pi/6$) has length $\sqrt{3}$, and
  - the side adjacent to $\pi/3$ (opposite to $\pi/6$) has length *1*
- It is thus straightforward to read off all six trig functions of either $\pi/6$ or $\pi/3$ (so long as one remembers that what is meant by "adjacent" and what is meant by "opposite" depends on which angle one is talking about). For example,

$$\begin{aligned} sin(\pi/3) &= opposite/hypoteneuse = (\sqrt{3})/2 \\ csc(\pi/3) &= hypoteneuse/opposite = 2/\sqrt{3} \\ cot(\pi/3) &= adjacent/opposite = 1/\sqrt{3},\ etc. \end{aligned} \tag{2}$$

and

$$\begin{aligned} sin(\pi/6) &= opposite'/hypoteneuse = 1/2 \\ csc(\pi/6) &= hypoteneuse/opposite' = 2/1 = 2 \\ cot(\pi/6) &= hypoteneuse/opposite' = (\sqrt{3})/1 = \sqrt{3},\ \ etc. \end{aligned} \tag{3}$$

where we have written $opposite'$ and $adjacent'$ for the sides opposite and adjacent to the angle $\pi/6$, with the primes serving as a reminder that the side opposite to $\pi/6$ is adjacent to $\pi/3$ and, similarly, the side adjacent to $\pi/6$ is opposite to $\pi/3$.

SOME HELP WITH REMEMBERING THE $\pi/6$, $\pi/3$ CASE

While it is easy to reconstruct an example of the right-angle triangle having internal angle $\pi/4$, so that there is no need to memorize the special $\pi/4$ triangle, the geometric construction needed to get an example of a triangle having non-right angles $\pi/6$ and $\pi/3$, though simple, involved a number of steps.

It may, thus, be worthwhile memorizing the example above of a triangle with non-right angles $\pi/6$ and $\pi/3$ This is actually fairly easy to do in a way that minimizes the amount of "cold memorization" needed, as follows:

- It is not hard to remember that such a triangle of the right size has one side of length *1*, one side of length $\sqrt{3}$ and one side of length *2* (especially once one notices that each of the first 3 positive integers occurs once, one with a square root, and takes into account that they three lengths have to be related by Pythagoras' Theorem)

- The rest is easy, and requires no memorization: the longest side, *2*, has to be the hypoteneuse (opposite the largest angle); the next longest side, $\sqrt{3}$ has to be opposite the larger of the two remaining angles, $\pi/3$, and the shortest side, *1*, has to be opposite the smallest of the angles, $\pi/6$

- Once one has the angles labelled and the sides labelled, reading off any of the six ratios for either $\pi/6$ or $\pi/3$ is straightforward; the figure gives you all 12 pieces of information at once

EXERCISES

(In the exercises below, all angles are given in radians)

1. Using Figure 2, determine the values of the three trig functions of the angle $\pi/4$ not already listed in Equations 1, namely $cos(\pi/4)$, $csc(\pi/4)$, and $cot(\pi/4)$.

2. Using Figure 3, determine the values of the three trig functions of the angle $\pi/3$ not already listed in Equations 2, namely $cos(\pi/3)$, $tan(\pi/3)$, and $sec(\pi/3)$.

3. Using Figure 3, determine the values of the three trig functions of the angle $\pi/6$ not already listed in Equations 3, namely $cos(\pi/6)$, $tan(\pi/6)$, and $sec(\pi/6)$.