1. Question 7.7.3 on page 316 of the text.

The problem involves the wave equation for a vibrating quarter circular membrane, where \( 0 \leq r \leq a \) and \( 0 \leq \theta \leq \pi/2 \), with \( u(r, \theta) = 0 \) everywhere on the boundary.

(a) Separation of variables \( u(r, \theta, t) = \phi(r, \theta)T(t) = \rho(r)\tau(\theta)T(t) \) leads to the eigenvalue problems

\[
\begin{align*}
  r^2 \rho'' + r \rho' + (\lambda r^2 - \mu) \rho &= 0 \quad (1) \\
  \tau'' &= -\mu \tau \quad (2) \\
  T'' &= -c^2 \lambda T \quad (3)
\end{align*}
\]

The solution to (3) is \( T(t) = A \cos(c\sqrt{\lambda}t) + B \sin(c\sqrt{\lambda}t) \); the frequency of this function, namely \( c\sqrt{\lambda} \), is the frequency of vibration.

The equations (1,2) determine the possible values of \( \lambda \) as follows. The boundary condition \( \tau(0) = \tau(\pi/2) = 0 \) implies that (2) has non-trivial solutions only of the form \( \tau(\theta) = b \sin(2m\theta) \), where \( \mu = (2m)^2 \), for \( m = 1, 2, 3, \ldots \). The only solutions to (1) for which \( \lim_{r \to 0^+} \rho(r) < \infty \) are Bessel functions of the form \( \rho(r) = J_{\sqrt{\mu}}(\sqrt{\lambda}r) = J_{2m}(\sqrt{\lambda}r) \). And the boundary condition \( \rho(a) = 0 \) implies that \( J_{2m}(\sqrt{\lambda}a) = 0 \). Letting \( z_{k,n} \) denote the successive zeros of \( J_k \) (with \( n = 1, 2, 3, \ldots \)), it follows that

\[
\sqrt{\lambda} = z_{2m,n}/a,
\]

where \( m, n = 1, 2, 3, \ldots \). So the possible frequencies of vibration are \( c z_{2m,n}/a \).

(b) From part (a), for each \( m, n \) there is a solution that satisfies the boundary conditions of the form

\[
  u_{m,n}(r, \theta, t) = J_{2m,n}(z_{2m,n}r/a) \sin(2m\theta) \left( A_{m,n} \cos(cz_{2m,n}t/a) + B_{m,n} \sin(cz_{2m,n}t/a) \right).
\]

So the most general possible solution is \( u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{m,n} \). The initial condition \( \partial u/\partial t(r, \theta, 0) = 0 \) implies that \( B_{m,n} = 0 \) for each \( m, n \). And applying Fourier sine series and Fourier-Bessel series to the remaining initial condition

\[
  u(r, \theta, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{m,n} J_{2m,n}(z_{2m,n}r/a) \sin(2m\theta) = g(r, \theta)
\]
yields that
\[ A_{m,n} = \frac{4}{\pi} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} g(r, \theta) J_{2m}(z_{2m,n}r/a) \sin(2m\theta) r \, dr \, d\theta \]
\[ \int_{0}^{a} (J_{2m}(z_{2m,n}r/a))^2 r \, dr \].

2. Solve the following first order equations for $u(x, t)$ using the method of characteristics.

(a) $2 \frac{\partial u}{\partial t} + 3 \frac{\partial u}{\partial x} = 0$

(b) $2 \frac{\partial u}{\partial t} + 3 \frac{\partial u}{\partial x} = 1$

(a) Note that if we write $u(x, t)$ (with the variables in that order) then $2 \frac{\partial u}{\partial t} + 3 \frac{\partial u}{\partial x} = \nabla u \cdot v$, where $v = (3, 2)$. This is a directional derivative of $u$ in the direction of $v$. Now, there are two ways to represent the line $\ell$ through a given point $(x_0, t_0)$ having direction $v$.

The line $\ell$ may also be represented as the set of all points of the form

\[(x, t) = (x_0 + 3s, t_0 + 2s) \text{ where } s \in \mathbb{R}. \]

Let $\omega(s) = u(x_0 + 3s, t_0 + 2s)$ be the restriction of $u$ to the line $\ell$. Then, by the chain rule,

\[ \omega'(s) = 2 \frac{\partial u}{\partial t}(x, t) + 3 \frac{\partial u}{\partial x}(x, t), \text{ where } (x, t) = (x_0 + 3s, t_0 + 2s). \]

Thus the original problem, for the given line $2x - 3t = c$, is equivalent to the equation $\omega'(s) = 0$. Integrating, one obtains the solution $\omega(s) = f(c)$, where $f$ is a constant depending on $c = 2x - 3t$. In terms of $u$, this is:

\[ u(x, t) = f(2x - 3t), \]

where $f$ is an arbitrary function. [Note: the solution may be checked directly, just by evaluating partial derivatives.]

(b) By (6), the equation is equivalent to $\omega'(s) = 1$, which has solution $\omega(s) = s + f(c)$, where, as before, $f$ is a constant depending on $c = 2x - 3t$. From (5), $s$ may be expressed as

\[ s = \frac{\langle (x, t) - (x_0, t_0), v \rangle}{\langle v, v \rangle} = \frac{1}{13} (3x + 2t) + \alpha, \text{ where } \alpha = -\frac{\langle (x_0, t_0), v \rangle}{\langle v, v \rangle}. \]

Since $(x_0, t_0)$ is an arbitrary point on the line $\ell$, it follows that $\alpha$, like $f$, is a constant depending on $c$ (i.e., depending on the line). Thus

\[ u(x, t) = s + f(c) = \frac{1}{13} (3x + 2t) + \alpha(2x - 3t) + f(2x - 3t) \]
\[ = \frac{1}{13} (3x + 2t) + g(2x - 3t), \]
where \( g = \alpha + f \) is an arbitrary function. [Note: again, this solution can be checked directly. Any solution that differs from this one by a function of \((2x - 3t)\) is also a solution.]

3. Solve the wave equation

\[
\frac{\partial^2 u}{\partial t^2} - 9 \frac{\partial^2 u}{\partial x^2} = 0 \quad -\infty < x < \infty
\]
on an infinite line, with initial conditions \( u(x, 0) = 0 \) and

\[
\frac{\partial u}{\partial t}(x, 0) = \begin{cases} 
1 & \text{if } |x| \leq 1 \\
0 & \text{if } |x| > 1 
\end{cases}
\]

Sketch the solution at times \( t = 0, 1, 2 \).

Write \( G(x) = \begin{cases} 
1 & \text{if } |x| \leq 1 \\
0 & \text{if } |x| > 1 
\end{cases} \)

Then, according to the d’Alembert solution,

\[
u(x, t) = \frac{1}{6} \int_{x-3t}^{x+3t} G(s) \, ds.
\]

Clearly \( u(x, 0) = 0 \). The integral can be calculated explicitly. For \( t = 1 \) and \( t = 2 \), it works out as follows.

\[
u(x, 1) = \begin{cases} 
(x + 4)/6 & -4 < x \leq -2 \\
1/3 & -2 < x \leq 2 \\
(4 - x)/6 & 2 < x < 4 \\
0 & \text{otherwise}
\end{cases}
\]

\[
u(x, 2) = \begin{cases} 
(x + 7)/6 & -7 < x \leq -5 \\
1/3 & -5 < x \leq 5 \\
(7 - x)/6 & 5 < x < 7 \\
0 & \text{otherwise}
\end{cases}
\]

4. (a) If \( u(x, t) \) satisfies the wave equation on the infinite line, show that \( u(x, -t) \) also satisfies the wave equation.

(b) If \( u(x, t) \) satisfies the heat equation on the infinite line, does \( u(x, -t) \) necessarily satisfy the heat equation? Explain.

Write \( \psi(x, t) = u(x, -t) \). Then \( \frac{\partial \psi}{\partial t}(x, t) = -\frac{\partial u}{\partial t}(x, -t) \) and \( \frac{\partial^2 \psi}{\partial t^2}(x, t) = \frac{\partial^2 u}{\partial t^2}(x, -t) \). And \( \frac{\partial^2 \psi}{\partial x^2}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, -t) \).

(a) Suppose \( u \) satisfies the wave equation. Then

\[
\frac{\partial^2 \psi}{\partial t^2}(x, t) - c^2 \frac{\partial^2 \psi}{\partial x^2}(x, t) = \frac{\partial^2 u}{\partial t^2}(x, -t) - c^2 \frac{\partial^2 u}{\partial x^2}(x, -t) = 0.
\]
So $\psi$ satisfies the wave equation.
(b) Suppose $u$ satisfies the heat equation. Then

$$\frac{\partial \psi}{\partial t}(x,t) - k \frac{\partial^2 \psi}{\partial x^2}(x,t) = -\frac{\partial u}{\partial t}(x,-t) - k \frac{\partial^2 u}{\partial x^2}(x,-t).$$

The latter is only zero if

$$-\frac{\partial u}{\partial t^2}(x,-t) - k \frac{\partial^2 u}{\partial x^2}(x,-t) = \frac{\partial u}{\partial t}(x,-t) - k \frac{\partial^2 u}{\partial x^2}(x,-t) \implies \frac{\partial u}{\partial t} = 0.$$

Thus only if $u$ is independent of $t$ (i.e. if $u$ is a steady state solution), will $\psi$ be a solution to the heat equation. In general, $\psi$ need not be a solution.

5. Question 10.3.1 on page 455 of the text.
(a) By definition

$$\mathcal{F}(c_1f + c_2g)(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (c_1f(x) + c_2g(x)) e^{-i\omega x} \, dx$$

$$= c_1 \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \right) + c_2 \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} \, dx \right)$$

$$= c_1 \mathcal{F}f(\omega) + c_2 \mathcal{F}g(\omega).$$

(b) Consider for example $f(x) = g(x) = e^{-x^2}$. Then

$$\mathcal{F}(fg)(\omega) = \frac{1}{\sqrt{8\pi}} e^{-\omega^2/8} \neq (\mathcal{F}f)(\omega)(\mathcal{F}g)(\omega) = \frac{1}{4\pi} e^{-\omega^2/2}.$$

6. Question 10.3.5. Note: the text and the lecture use opposite signs in the exponent of the Fourier transform. Using the convention of the lecture, if $F = \mathcal{F}f$, then

$$\mathcal{F}^{-1} \left( e^{i\omega \beta} F(\omega) \right)(x) = \int_{-\infty}^{\infty} e^{i\omega \beta} F(\omega) e^{i\omega x} \, d\omega$$

$$= \int_{-\infty}^{\infty} F(\omega) e^{i\omega(x+\beta)} \, d\omega$$

$$= (\mathcal{F}^{-1} F)(x + \beta)$$

$$= f(x + \beta).$$

[The convention used in the text results in $f(x - \beta)$ instead.]