FROM LAX MONAD EXTENSIONS TO TOPOLOGICAL THEORIES

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To Manuela, teacher and friend

Abstract. We investigate those lax extensions of a Set-monad $\mathbb{T} = (T, m, e)$ to the category V-Rel of sets and V-valued relations for a quantale $\mathsf{V} = (V, \otimes, k)$ that are fully determined by maps $\xi : TV \to V$. We pay special attention to those maps ξ that make V a $\mathbb{T}$-algebra and, in fact, $(V, \otimes, k)$ a monoid in the category $\mathsf{Set}^{\mathbb{T}}$ with its cartesian structure. Any such map ξ forms the main ingredient to Hofmann's notion of topological theory.

Introduction

The lax-algebraic setting, originally considered in [5] and [4] as a common syntax for the categories of lax algebras discussed in [2], was generalized by Seal in [9] and in this form adopted in [7] and studied by various authors. A very powerful specialization of the lax-algebraic setting was introduced by Hofmann [6] in the form of his topological theories, which in particular cover Barr's presentation of topological spaces [1] and the Clementino-Hofmann presentation of approach spaces (see [2, 7]). This paper carefully studies how the Hofmann notion may be characterized within the Seal setting.

Recall that, for an endofunctor T of sets and maps and a (commutative and unital) quantale $\mathsf{V} = (V, \otimes, k)$, Seal considers lax functors $\hat{T}$ of sets and V-valued relations (or "matrices") which, when applied to maps or their opposites, will generally increase the value in the pointwise order of their hom-sets in comparison to an application by T; furthermore, if T carries a monad structure, $\hat{T}$ is said to laxly extend the monad if the unit and the multiplication of the monad become oplax transformations when T is replaced by $\hat{T}$. In Hofmann's

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setting, such lax extensions get constructed from a mere function $\xi : TV \to V$ satisfying certain compatibility conditions with the monad and the quantale V (see also [8]). One of their special properties not available in the general Seal environment is that these lax extensions are *dualizing*, i.e., they are invariant under the involution given by inverting V-relations.

But being dualizing does not characterize Hofmann's lax monad extensions among all. In this paper we identify a stronger property, called *algebraicity* here, as the key ingredient to a property-based characterization of the Hofmann extensions amongst all lax extensions of a Set-monad satisfying the so called Beck-Chevalley property (BC). Algebraicity ensures that, given a V-relation $r : X \rightarrow\!\!\!\!\!\!+\;\, Y$, which may be equivalently described as a V-relation $\tilde{r} : X \times Y \rightarrow\!\!\!\!\!\!+\;\, 1$, the value of $\hat{T}r$ may actually be recovered from $\hat{T}\tilde{r}$. Such lax extensions are necessarily induced by a single map $\xi : TV \to V$ the additional properties of which are shown to correspond to known properties of the induced lax extension. Our central result establishes a 1-1 correspondence between algebraic lax extensions of a given Set-functor T satisfying BC and monotone maps ξ that are laxly compatible with the monoid structure of V (Theorem 1.4.4). Hofmann's theories, whether in their lax or strict forms, are shown to grow naturally out of this basic correspondence (Theorems 1.6.2 and 2.2.1).

While many of the key ideas and techniques of the proofs presented here are already present in [6], our presentation within the Seal context is new. We have also tried to minimize the use of "elementwise calculations"; in fact, most of our extensive calculations use exclusively the compositional structure of the monoidal-closed category V-Rel of sets and V-relations with its order enrichment.

1. Algebraic lax extensions

1.1 The symmetric monoidal-closed quantaloid V-Rel. Let $\mathsf{V} = (V, \otimes, k)$ be a quantale[1] which, for simplicity, is always assumed to be commutative. One associates with V the quantaloid[2] V-Rel of sets and V-relations[3]; here a

[1]A *quantale* is a complete lattice with a monoid structure such that the binary operation $\otimes$ distributes over arbitrary suprema in each variable. The neutral element k may be smaller than the top element $\top$ but it is always assumed to be larger than the bottom element $\bot$.

[2]A *quantaloid* is a Sup-enriched category, where Sup is the monoidal-closed category of complete lattices and sup-preserving maps.

[3]A V-relation $r : X \rightarrow\!\!\!\!\!\!+\;\, Y$ may have different set-theoretic representations. For example, for $\mathsf{V} = \mathbf{2}$ the two-element chain, r is usually represented by a subset of $X \times Y$. The representation that matters in this paper is that of a function $X \times Y \to \mathsf{V}$ for which we introduce the notation $\overrightarrow{r}$ in (1.1.i) below whenever we want to emphasize its role as an arrow in Set in this specific form, rather than as a morphism in V-Rel.

V-relation from a set X to a set Y is given by a function $X \times Y \to \mathsf{V}$, written as $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, and composition with $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ is defined by

$$s \cdot r(x,z) = \bigvee_{y \in Y} r(x,y) \otimes s(y,z).$$

When considering V as a one-object quantaloid, we obtain a homomorphism

$$\mathsf{V}^{\mathrm{op}} \to \mathsf{V}\text{-}\mathsf{Rel},\ v \mapsto (v : 1 \relbar\joinrel\mapstochar\joinrel\rightarrow 1),$$

which embeds V^{op} fully into $\mathsf{V}\text{-}\mathsf{Rel}$. There is an obvious isomorphism

$$\mathsf{V}\text{-}\mathsf{Rel}^{\mathrm{op}} \to \mathsf{V}\text{-}\mathsf{Rel},\ (r : X \to Y) \mapsto (r^\circ : Y \relbar\joinrel\mapstochar\joinrel\rightarrow X) \text{ with } r^\circ(y,x) = r(x,y),$$

of quantaloids, which makes $\mathsf{V}\text{-}\mathsf{Rel}$ self-dual. There is also a faithful functor

$$(-)_\circ : \mathsf{Set} \to \mathsf{V}\text{-}\mathsf{Rel},\ (f : X \to Y) \mapsto (f = f_\circ : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y)$$

with $f_\circ(x,y) = \left\{ \begin{array}{ll} k & \text{if } f(x) = y, \\ \bot & \text{else} \end{array} \right\}$, whose opposite we may compose with the above functor $(r \mapsto r^\circ)$ to obtain

$$(-)^\circ : \mathsf{Set}^{\mathrm{op}} \to \mathsf{V}\text{-}\mathsf{Rel},\ (f : X \to Y) \mapsto (f^\circ : Y \relbar\joinrel\mapstochar\joinrel\rightarrow X).$$

Note that in $\mathsf{V}\text{-}\mathsf{Rel}$, considered as a 2-category, one has $f_\circ \dashv f^\circ$ for every map f in Set. The functor $(-)_\circ$ has a right adjoint

$$\mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{Set},\ (r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y) \mapsto (r^\sharp : V^X \to V^Y),$$

with $r^\sharp(\varphi)(y) = \bigvee_{x \in X} \varphi(x) \otimes r(x,y)$ for all $\varphi \in V^X$, $y \in Y$. This functor is represented by 1 and induces the V-powerset monad on Set (see 1.5.2 below). Here we are only interested in the counits of this adjunction,

$$\iota_X : V^X \relbar\joinrel\mapstochar\joinrel\rightarrow X,\ \iota_X(\varphi, x) = \varphi(x),$$

in particular in

$$\iota : V \cong V^1 \xrightarrow{\iota_1} 1,\ \iota(v) = v.$$

When we write a V-relation $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ equivalently as $\tilde{r} : X \times Y \relbar\joinrel\mapstochar\joinrel\rightarrow 1$, the couniversal property of ι makes $\tilde{r}$ factor uniquely through the map $\overrightarrow{r} : X \times Y \to V$:

(1.1.i)
$$\begin{tikzcd} V \arrow[r, "\iota", "\shortmid"{marking}] & 1 \\ X \times Y \arrow[u, "\overrightarrow{r}"] \arrow[ur, "\tilde{r}"', "\shortmid"{marking}] & \end{tikzcd}$$

This trivial observation is of great importance to our further study of lax extensions $\hat{T} : \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{V}\text{-}\mathsf{Rel}$ of a given functor $T : \mathsf{Set} \to \mathsf{Set}$ as defined below.

In fact, the extensions of interest will preserve the composition of a V-relation preceded by a Set-map strictly, so that

$$\hat{T}\tilde{r} = \hat{T}\iota \cdot T(\overrightarrow{r}).$$

Consequently, the value of $\hat{T}\tilde{r}$ is already determined by $\hat{T}\iota$ and the given functor T, a fact that we will exploit shortly.

With $X \otimes Y = X \times Y$ and

$$r \otimes r' : X \times X' \relbar\joinrel\mapstochar\joinrel\rightarrow Y \times Y',\ r \otimes r'((x,x'),(y,y')) = r(x,y) \otimes r'(x',y'),$$

for $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $r' : X' \relbar\joinrel\mapstochar\joinrel\rightarrow Y'$, we obtain a 2-functor

$$(-) \otimes (-) : \mathsf{V}\text{-}\mathsf{Rel} \times \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{V}\text{-}\mathsf{Rel},$$

which preserves suprema in each variable and makes V-Rel a symmetric monoidal category whose tensor product is compatible with the ordered structure (in fact, with the Sup-enrichment). Moreover, since the natural isomorphisms

$$\text{(1.1.ii)} \qquad \mathsf{V}\text{-}\mathsf{Rel}(X,Y) \xrightarrow{\sim} \mathsf{V}\text{-}\mathsf{Rel}(X \times Y, 1),\ r \mapsto \tilde{r},$$

used above trivially generalize to natural isomorphisms

$$\mathsf{V}\text{-}\mathsf{Rel}(X, Y \times Z) \xrightarrow{\sim} \mathsf{V}\text{-}\mathsf{Rel}(X \times Y, Z),$$

so that $(-) \otimes Y \dashv Y \otimes (-)$, V-Rel is actually monoidal closed, with the internal hom being given by the tensor product again.

Our up-coming treatment of lax extensions of monads from Set to V-Rel entails intensive use of compatibility rules of the passages from r to r°, $\tilde{r}$, $\overrightarrow{r}$ with the various structures of V-Rel. We summarize them in the following Proposition, the proof of which may be left as a straightforward exercise.

1.1.1. **Proposition.** *For maps $f : X \to Y$, $f' : X' \to Y'$, $g : Z \to Y$, $h : Y \to Z$, $j : Y \to X$, and V-relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $r' : X' \relbar\joinrel\mapstochar\joinrel\rightarrow Y'$, $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$, one has:*

(1) $\overrightarrow{r^\circ} = \overrightarrow{r} \cdot \sigma_{Y,X}$, $\widetilde{r^\circ} = \tilde{r} \cdot \sigma_{Y,X}$;
(2) $\iota \cdot \overrightarrow{1_X} = !_X \cdot \delta_X^\circ = \widetilde{1_X}$;
(3) $\overrightarrow{s \cdot f} = \overrightarrow{s} \cdot (f \times 1_Z)$, $\widetilde{s \cdot f} = \tilde{s} \cdot (f \times 1_Z)$;
(4) $\overrightarrow{g^\circ \cdot r} = \overrightarrow{r} \cdot (1_X \times g)$, $\widetilde{g^\circ \cdot r} = \tilde{r} \cdot (1_X \times g)$;
(5) $\iota \cdot \overrightarrow{h \cdot r} = \iota \cdot \overrightarrow{r} \cdot (1_X \times h)^\circ$, $\widetilde{h \cdot r} = \tilde{r} \cdot (1_X \times h)^\circ$;
(6) $\iota \cdot \overrightarrow{s \cdot j^\circ} = \iota \cdot \overrightarrow{s} \cdot (j \times 1_Z)^\circ$, $\widetilde{s \cdot j^\circ} = \tilde{s} \cdot (j \times 1_Z)^\circ$;
(7) $\iota \cdot \overrightarrow{s \cdot r} = \iota \cdot \otimes \cdot (\overrightarrow{r} \times \overrightarrow{s}) \cdot (1_X \times \delta_Y \times 1_Z) \cdot (1_X \times !_Y \times 1_Z)^\circ$,
$\widetilde{s \cdot r} = (\tilde{r} \otimes \tilde{s}) \cdot (1_X \times \delta_Y \times 1_Z) \cdot (1_X \times !_Y \times 1_Z)^\circ$;
(8) $\overrightarrow{\iota} = 1_V$, $\tilde{\iota} = \iota$, $\overrightarrow{\iota \otimes \iota} = \otimes$, $\iota \cdot k = 1_1$, $\widetilde{\iota \otimes \iota} = \iota \otimes \iota = \iota \cdot \otimes$;
(9) $(f \times f')_\circ = f_\circ \otimes f'_\circ$, $(f \times f')^\circ = f^\circ \otimes (f')^\circ$; $(r \otimes r')^\circ = r^\circ \otimes (r')^\circ$;

(10) $\overrightarrow{r \otimes r'} = \otimes\cdot(\overrightarrow{r} \times \overrightarrow{r'})\cdot(1_X \times \sigma_{X',Y} \times 1_{Y'})$, $\widetilde{r \otimes r'} = (\tilde{r} \otimes \tilde{r'})\cdot(1_X \times \sigma_{X',Y} \times 1_{Y'})$.

(Here $!_X : X \to 1$, $\delta_X : X \to X \times X$, $\sigma_{X,Y} : X \times Y \to Y \times X$, $k : 1 \to V$, $\otimes : V \times V \to V$ have the obvious meaning. We identify X, $X \times 1$, $1 \times X$ and disregard all associativity isomorphisms for $\times$.)

Note that (9) implies that $(-)_\circ : \mathsf{Set} \to \mathsf{V\text{-}Rel}$, $(-)^\circ : \mathsf{Set}^{\mathrm{op}} \to \mathsf{V\text{-}Rel}$ are homomorphisms of monoidal categories when Set is provided with its cartesian structure; furthermore, $\mathsf{V\text{-}Rel}$ is selfdual, not only as a Sup-enriched category, but also as a monoidal category.

1.2 Algebraic lax extensions. Recall from [7, 9] the following definition:

1.2.1. **Definition.** Given a Set-functor T, a *lax extension* $\hat{T}$ of T to $\mathsf{V\text{-}Rel}$ is a lax functor $\hat{T} : \mathsf{V\text{-}Rel} \to \mathsf{V\text{-}Rel}$ that coincides with T on objects and satisfies

$$\begin{tikzcd} \mathsf{V\text{-}Rel} \arrow[r, "\hat{T}"] & \mathsf{V\text{-}Rel} \\ \mathsf{Set} \arrow[u, "(-)_\circ"] \arrow[r, "T"'] \arrow[ur, phantom, "\geq"] & \mathsf{Set} \arrow[u, "(-)_\circ"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{V\text{-}Rel} \arrow[r, "\hat{T}"] & \mathsf{V\text{-}Rel} \\ \mathsf{Set} \arrow[u, "(-)^\circ"] \arrow[r, "T"'] \arrow[ur, phantom, "\geq"] & \mathsf{Set} \arrow[u, "(-)^\circ"'] \end{tikzcd}$$

Hence, $\hat{T}$ is given by functions

$$\hat{T}_{X,Y} : \mathsf{V\text{-}Rel}(X,Y) \to \mathsf{V\text{-}Rel}(TX,TY) \quad (X, Y \in \mathsf{Set})$$

such that

(1) $r \leq r' \implies \hat{T}r \leq \hat{T}r'$,
(2) $\hat{T}s \cdot \hat{T}r \leq \hat{T}(s \cdot r)$,
(3) $Tf \leq \hat{T}f$ and $(Tf)^\circ \leq \hat{T}(f^\circ)$,

for all sets X, Y, Z, V-relations $r, r' : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ and maps $f : X \to Y$.

As shown in [7, 9], these conditions imply that, respectively, composition with maps from the right and with converses of maps from the left are preserved strictly:

$$\hat{T}(s \cdot f) = \hat{T}s \cdot Tf \text{ and } \hat{T}(g^\circ \cdot r) = (Tg)^\circ \cdot \hat{T}r$$

(where $g : Z \to Y$ is a map). The lax extension $\hat{T}$ is *flat* if $\hat{T}1_X = 1_{TX}$; because of the whiskering properties, the lax-commutative diagrams above commute strictly in this case.

If T is the carrier of a monad $\mathbb{T} = (T, m, e)$, one says that $\hat{\mathbb{T}} = (\hat{T}, m, e)$ is a *lax extension* of $\mathbb{T}$ to $\mathsf{V\text{-}Rel}$ if the lax extension of the functor T makes $m : \hat{T}\hat{T} \to \hat{T}$ and $e : 1_{\mathsf{V\text{-}Rel}} \to \hat{T}$ oplax natural transformations. Hence, in addition to (1)-(3) above, one must have

(4) $m_Y \cdot \hat{T}\hat{T}r \leq \hat{T}r \cdot m_X$,

(5) $e_Y \cdot r \leq \hat{T}r \cdot e_X$

for all V-relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$. With the adjunctions $m_X \dashv m_X^\circ$ and $e_X \dashv e_X^\circ$ (for all X), these conditions may be written equivalently as

(4°) $\hat{T}\hat{T}r \cdot m_X^\circ \leq m_Y^\circ \cdot \hat{T}r$,

(5°) $r \cdot e_X^\circ \leq e_Y^\circ \cdot r$.

Let us now consider a lax extension $\hat{T} : \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{V}\text{-}\mathsf{Rel}$ of a Set-functor T and check on its compatibility with the natural isomorphisms (1.1.ii). More concretely, we wish to study lax extensions $\hat{T}$ that allow us to naturally recover $\hat{T}r$ from $\hat{T}\tilde{r}$, for every V-relation $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, and therefore consider the diagram (1.2.i)

$$\begin{tikzcd}
 & & \mathsf{V}\text{-}\mathsf{Rel}(X,Y) \arrow[dll, "\widetilde{(-)}"'] \arrow[drr, "\hat{T}_{X,Y}"] & & \\
\mathsf{V}\text{-}\mathsf{Rel}(X\times Y,1) \arrow[dr, "\hat{T}_{X\times Y,1}"'] & & & & \mathsf{V}\text{-}\mathsf{Rel}(TX,TY) \arrow[dl, "\widetilde{(-)}"] \\
 & \mathsf{V}\text{-}\mathsf{Rel}(T(X\times Y),T1) \arrow[rr, "\mathsf{V}\text{-}\mathsf{Rel}(\mathrm{can}^\circ_{X,Y},\,!_{T1})"'] & & \mathsf{V}\text{-}\mathsf{Rel}(TX\times TY,1) &
\end{tikzcd}$$

with $\mathrm{can}_{X,Y} = \langle T\pi_1, T\pi_2\rangle : T(X\times Y) \to TX \times TY$.

1.2.2. **Definition.** The lax extension $\hat{T}$ of T is *algebraic* if diagram (1.2.i) commutes for all X, Y, that is: if

$$\widetilde{\hat{T}r} = (TX \times TY \overset{\mathrm{can}^\circ}{\relbar\joinrel\mapstochar\joinrel\rightarrow} T(X\times Y) \overset{\hat{T}\tilde{r}}{\relbar\joinrel\mapstochar\joinrel\rightarrow} T1 \xrightarrow{!_{T1}} 1) \tag{1.2.ii}$$

for all $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$.

Hence, if $\hat{T}$ is algebraic, the value of $\hat{T}r$ may be obtained from $\hat{T}\tilde{r}$ in a natural way, despite the fact that the maps $\mathrm{can}_{X,Y}$ and $!_{T1}$ will generally "lose information", i.e., will generally not be injective. If they are injective (and especially if T preserves finite products, so that $\mathrm{can}_{X,Y}$ and 1_{T1} are bijective), then also the map $\mathsf{V}\text{-}\mathsf{Rel}(\mathrm{can}^\circ_{X,Y},\,!_{T1})$ is injective and $\hat{T}\tilde{r}$ may be recovered from $\hat{T}r$, as $\hat{T}\tilde{r} = !^\circ_{T1} \cdot \widetilde{\hat{T}r} \cdot \mathrm{can}_{X,Y}$.

Before discussing examples, let us state some immediate facts.

1.2.3. **Proposition.** (1) *A lax extension $\hat{T}$ of T is algebraic if, and only if, there is a map $\xi : TV \to V$ with*

$$\widetilde{\hat{T}r} = \iota \cdot \xi \cdot T\overrightarrow{r} \cdot \mathrm{can}^\circ_{X,Y} \tag{1.2.iii}$$

for all $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$.

(2) *Any map* ξ *with* (1.2.iii) *for all* r *must make the diagram*

(1.2.iv)
$$\begin{tikzcd} TV \arrow[r, "\hat{T}\iota", "\shortmid"{marking}] \arrow[d, "\xi"'] & T1 \arrow[d, "!_{T1}"] \\ V \arrow[r, "\iota", "\shortmid"{marking}] & 1 \end{tikzcd}$$

commutative and is therefore uniquely determined.

(3) *An algebraic lax extension* $\hat{T}$ *is* dualizing, *that is:* $\hat{T}(r^\circ) = (\hat{T}r)^\circ$ *for all* r. *Moreover, if the* $\otimes$*-neutral element* k *is the top element in* V, $\hat{T}$ *is flat on* $!_X : X \to 1$, *that is:* $\hat{T}\,!_X = T\,!_X$ *for all sets* X.

Proof. (1),(2): Since ι is the counit for the left-adjoint functor $(-)_\circ : \mathsf{Set} \to \mathsf{V}\text{-}\mathsf{Rel}$ (see 1.1), we may define ξ by (1.2.iv), i.e., $\xi = \overrightarrow{1_{T1} \cdot \hat{T}\iota}$. If $\hat{T}$ is algebraic, from $\tilde{r} = \iota \cdot \overrightarrow{r}$ we obtain $\hat{T}\tilde{r} = \hat{T}\iota \cdot T\overrightarrow{r}$ and then

$$\widetilde{\hat{T}r} = !_{T1} \cdot \hat{T}\tilde{r} \cdot \mathrm{can}^\circ_{X,Y} = !_{T1} \cdot \hat{T}\iota \cdot T\overrightarrow{r} \cdot \mathrm{can}^\circ_{X,Y} = \iota \cdot \xi \cdot T\overrightarrow{r} \cdot \mathrm{can}^\circ_{X,Y}. \tag{1.2.v}$$

Conversely, exploiting (1.2.iii) for $r = \iota$ gives

$$\widetilde{\hat{T}\iota} = \iota \cdot \xi \cdot \mathrm{can}^\circ_{V,1}$$

since $\overrightarrow{\iota} = 1_V$. With $\mathrm{can}_{V,1} = \langle 1_{TV}, T\,!_V \rangle : TV \to TV \times T1$ one obtains for all $\mathfrak{v} \in TV$, $\mathfrak{a} \in T1$

$$\hat{T}\iota(\mathfrak{v}, \mathfrak{a})) = \begin{cases} \xi(\mathfrak{v}) & \text{if } \mathfrak{a} = T\,!_V(\mathfrak{v}), \\ \bot & \text{else,} \end{cases} \tag{1.2.vi}$$

and then

$$(\,!_{TV} \cdot \hat{T}\iota)(\mathfrak{v}) = \bigvee_{\mathfrak{a} \in T1} \hat{T}\iota(\mathfrak{v}, \mathfrak{a}) = \xi(\mathfrak{v}) = (\iota \cdot \xi)(\mathfrak{v}).$$

Hence, (1.2.iv) commutes, and algebraicity of $\hat{T}$ follows from interchanging the computational steps in (1.2.v).

(3)

$$\begin{aligned} \widetilde{\hat{T}(r^\circ)} &= !_{T1} \cdot \hat{T}(\widetilde{r^\circ}) \cdot \mathrm{can}^\circ_{X,Y} \\ &= !_{T1} \cdot \hat{T}\tilde{r} \cdot T\sigma_{Y,X} \cdot \mathrm{can}^\circ_{X,Y} \\ &= !_{T1} \cdot T\tilde{r} \cdot \mathrm{can}^\circ_{X,Y} \cdot \sigma_{TY,TX} \\ &= \widetilde{\hat{T}r} \cdot \sigma_{TY,TX} \\ &= \widetilde{(\hat{T}r)^\circ}, \end{aligned}$$

for all $r : X \to\!\!\!\!\!\!\mapsto Y$. Writing temporarily $u : 1 \to 1$ for the identity morphism on the singleton set 1, one has (with the identification $1 \times 1 = 1$) $\tilde{u} = u$, hence

$$\widetilde{\hat{T}u} = !_{T1} \cdot \hat{T}u \cdot \mathrm{can}^{\circ}_{1,1} = !_{T1} \cdot \hat{T}u \cdot \delta^{\circ}_{T1}.$$

Consequently, $\hat{T}u(\mathfrak{a}, \mathfrak{b}) = \bot$ whenever $\mathfrak{a} \neq \mathfrak{b}$ in $T1$, and since $Tu \leq \hat{T}u$ we conclude $\hat{T}u = Tu$ if $k = \top$ in V. Now, for all X,

$$\hat{T}\,!_X = \hat{T}(u \cdot !_X) = \hat{T}u \cdot T\,!_X = Tu \cdot T\,!_X = T\,!_X$$

follows. □

1.2.4. **Examples.** (1) The Barr extension $\overline{T}$ of T to $\mathsf{Rel} = 2\text{-}\mathsf{Rel}$ (see [7, 2]) is algebraic. Indeed, when representing a relation $r : X \to\!\!\!\!\!\!\mapsto Y$ as a span $(X \overset{r_1}{\leftarrow} R \overset{r_2}{\to} Y)$ we obtain $\tilde{r}$ represented by $(X \times Y \overset{i}{\leftarrow} R \overset{!_R}{\to} 1)$ with $i = \langle r_1, r_2 \rangle$ the inclusion map, so that $\overline{T}r = Tr_2 \cdot (Tr_1)^{\circ}$ and $\overline{T}\tilde{r} = T\,!_R \cdot (Ti)^{\circ}$, by definition of the Barr extension $\overline{T}$. But

$$\begin{tikzcd}
T(X \times Y) \arrow[d, "\mathrm{can}_{X,Y}"'] & TR \arrow[l, "Ti"'] \arrow[r, "T\,!_R"] \arrow[dl, "j"] \arrow[dr, "!_{TR}"] & T1 \arrow[d, "!_{T1}"] \\
TX \times TY & & 1
\end{tikzcd}$$

with $j := \langle Tr_1, Tr_2 \rangle$ commutes, which shows $\widetilde{\overline{T}r} = !_{T1} \cdot \overline{T}\tilde{r} \cdot \mathrm{can}^{\circ}_{X,Y}$.

(2) The Kleisli extension $\check{T}$ of T to Rel (see [7, 9]) generally fails to be algebraic. For example, for the powerset functor P, $\check{P}1_X$ is the inclusion relation on PX which, even for $X = 1$, is not the identity relation. Consequently, by 1.2.3(3), $\check{P}$ cannot be algebraic; likewise for the Kleisli extension $\check{F}$ of the filter functor F (see [7, 9]).

1.3 The structure map of an algebraic lax extension. For an algebraic lax extension $\hat{T}$ of a Set-functor T we call the map $\xi : TV \to V$ with (1.2.iii) the *structure map* of $\hat{T}$. Our goal is to identify necessary properties of the map ξ which, in turn, will make $\hat{T}$ (when defined by 1.2.iii) a lax extension of T.

Let us first observe that when $\mathsf{Set}(X, V)$ is provided with the pointwise order of V, there are order isomorphisms

$$\mathsf{Set}(X, V) \to \mathsf{V}\text{-}\mathsf{Rel}(X, 1), \qquad \varphi \mapsto \iota \cdot \varphi$$

$$\begin{array}{ccccc}
\mathsf{V}\text{-}\mathsf{Rel}(X, Y) & \to & \mathsf{V}\text{-}\mathsf{Rel}(X \times Y, 1) & \to & \mathsf{Set}(X \times Y, V). \\
r & \mapsto & \tilde{r} & \mapsto & \overrightarrow{r}
\end{array}$$

1.3.1. **Proposition.** *The structure map ξ of an algebraic lax extension $\hat{T}$ of a* Set*-functor T satisfies*

$$k\cdot !_{T1} \leq \xi \cdot Tk \text{ and } \otimes \cdot \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle \leq \xi \cdot T\otimes .$$

(1.3.i)
$$\begin{tikzcd} T1 \arrow[r, "Tk"] \arrow[d, "!_{T1}"'] \arrow[dr, phantom, "\leq"] & TV \arrow[d, "\xi"] \\ 1 \arrow[r, "k"'] & V \end{tikzcd} \qquad \begin{tikzcd} T(V\times V) \arrow[r, "T\otimes"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle"'] \arrow[dr, phantom, "\leq"] & TV \arrow[d, "\xi"] \\ V\times V \arrow[r, "\otimes"] & V \end{tikzcd}$$

Proof. Exploiting the commutativity of

$$\begin{tikzcd} X \arrow[r, "\delta_X"] \arrow[d, "!_X"'] & X\times X \arrow[d, "\overrightarrow{1_X}"] \\ 1 \arrow[r, "k"'] & V \end{tikzcd}$$

for first $X = T1$ and then $X = 1$, and writing u for the identity map of 1, we obtain from $Tu \leq \hat{T}u$ the first of the two claims:

$$\begin{aligned} \iota\cdot k\cdot !_{T1} &= \iota\cdot \overrightarrow{1_{T1}}\cdot \delta_{T1} = \iota\cdot \overrightarrow{Tu}\cdot \delta_{T1} \leq \iota\cdot \overrightarrow{\hat{T}u}\cdot \delta_{T1} = \widetilde{\hat{T}u}\cdot \delta_{T1} \\ &= \iota\cdot \xi\cdot T\overrightarrow{u}\cdot \mathrm{can}^\circ_{1,1}\cdot \delta_{T1} = \iota\cdot \xi\cdot T\overrightarrow{u}\cdot T\delta_1 = \iota\cdot\xi\cdot Tk. \end{aligned}$$

In order to show the second inequality, we need the auxiliary inequality

$$(\hat{T}\iota \otimes 1_{TV})\cdot \mathrm{can}_{V,V} \leq \mathrm{can}_{1,V}\cdot \hat{T}(\iota\otimes 1_V) \tag{1.3.ii}$$

which is proved afterwards. With (1.3.ii) and the lax functoriality of $\hat{T}$ we then obtain:

$$\begin{aligned} \iota\cdot\otimes\cdot\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle &= (\iota\otimes\iota)\cdot(\xi\times\xi)\cdot \mathrm{can}_{V,V} \\ &= ((\iota\cdot\xi)\otimes(\iota\cdot\xi))\cdot \mathrm{can}_{V,V} \\ &= ((\,!_{T1}\cdot\hat{T}\iota)\otimes(\,!_{T1}\cdot\hat{T}\iota))\cdot \mathrm{can}_{V,V} \\ &= !_{T1\times T1}\cdot(\hat{T}\iota\otimes\hat{T}\iota)\cdot \mathrm{can}_{V,V} \\ &= !_{T1\times T1}\cdot(1_{T1}\otimes\hat{T}\iota)\cdot(\hat{T}\iota\otimes 1_{TV})\cdot \mathrm{can}_{V,V} \\ &\leq !_{T1\times T1}\cdot(1_{T1}\otimes\hat{T}\iota)\cdot\langle T\,!_V, 1_{TV}\rangle\cdot\hat{T}(\iota\otimes 1_V) \\ &= !_{T1}\cdot\hat{T}\iota\cdot\hat{T}(\iota\otimes 1_V) && (*) \\ &\leq !_{T1}\cdot\hat{T}(\iota\cdot(\iota\otimes 1_V)) \\ &= !_{T1}\cdot\hat{T}(\iota\cdot\otimes) \\ &= !_{T1}\cdot\hat{T}\iota\cdot T\otimes = \iota\cdot\xi\cdot T\otimes; \end{aligned}$$

here step $(*)$ follows from the easily established fact that

$$!_{Z\times Y}\cdot(1_Z\otimes r)\cdot\langle f, 1_X\rangle = !_Y\cdot r$$

holds for all $r : X \rightarrow\!\!\!\!\!\!\mapsto Y$ and $f : X\to Z$.

Attending now to (1.3.ii), let us first note that with 1.1.1 one has

$$\widetilde{\iota\otimes 1_V} = \tilde{\iota}\otimes\widetilde{1_V} = \iota\otimes(\,!_V\cdot\delta_V^\circ) = \iota\cdot(1_V\times !_V)\cdot(1_V\times\delta_V)^\circ = \iota\cdot\pi_1\cdot(1_V\times\delta_V)^\circ,$$

with $\pi_1 : V\times V\to V$. Consequently, since with $s := \iota\cdot\pi_1$ and $j := (1_V\times\delta_V)$ one has $\hat{T}s = \hat{T}(s\cdot j^\circ\cdot j) = \hat{T}(s\cdot j^\circ)\cdot Tj$ and, hence, $\hat{T}s\cdot(Tj)^\circ \leq \hat{T}(s\cdot j^\circ)$, we obtain

$$\hat{T}\iota\cdot T\pi_1\cdot T(1_V\times\delta_V)^\circ \leq \hat{T}(\widetilde{\iota\otimes 1_V}).$$

This gives

$$\begin{array}{rcl} \hat{T}(\widetilde{\iota\otimes 1_V}) & = & !_{T1}\cdot\hat{T}(\widetilde{\iota\otimes 1_V})\cdot\mathrm{can}^\circ_{V\times V,V} \\ & \geq & !_{T1}\cdot\hat{T}\iota\cdot T\pi_1\cdot T(1_V\times\delta_V)^\circ\cdot\mathrm{can}^\circ_{V\times V,V} \\ & = & \iota\cdot\xi\cdot T\pi_1\cdot\langle 1_{T(V\times V)}, T\pi_2\rangle^\circ \end{array}$$

and, when we evaluate the last term at $(\mathfrak{w},\mathfrak{v})\in T(V\times V)\times TV$,

$$(\hat{T}(\iota\otimes 1_V))(\mathfrak{w},\mathfrak{v}) \geq \begin{cases}\xi(T\pi_1(\mathfrak{w})) & \text{if } T\pi_2(\mathfrak{w})=\mathfrak{v},\\ \bot & \text{else}\end{cases} \tag{1.3.iii}$$

Consequently, since $\mathrm{can}_{1,V} = \langle T!_V, 1_{TV}\rangle$, with $(\mathfrak{a},\mathfrak{u})\in T1\times TV$ one has on the one hand

$$(\mathrm{can}_{1,V}\cdot\hat{T}(\iota\otimes 1_V))(\mathfrak{w},(\mathfrak{a},\mathfrak{u})) \geq \begin{cases}\xi(T\pi_1(\mathfrak{w})) & \text{if } T\pi_2(\mathfrak{w})=\mathfrak{u} \text{ and } T\,!_{V\times V}(\mathfrak{w})=\mathfrak{a}\\ \bot & \text{else.}\end{cases}$$

On the other hand one easily sees that the right-hand side of the last inequality is precisely the value of

$$\begin{array}{rcl} ((\hat{T}\iota\otimes 1_V)\cdot\mathrm{can}_{V,V})(\mathfrak{w},(\mathfrak{a},\mathfrak{u})) & = & (\hat{T}\iota\otimes 1_{TV})((T\pi_1(\mathfrak{w}),T\pi_2(\mathfrak{w})),(\mathfrak{a},\mathfrak{u})) \\ & = & \hat{T}\iota(T\pi_1(\mathfrak{w}),\mathfrak{a})\otimes 1_{TV}(T\pi_2(\mathfrak{w}),\mathfrak{u}) \end{array}$$

when one takes (1.2.vi) into account. $\square$

1.4 Obtaining a lax extension from a structure map. Given a map $\xi : TV\to V$ we may define a family of maps

$$\hat{T}_{X,Y} : \mathsf{V}\text{-}\mathsf{Rel}(X,Y)\to\mathsf{V}\text{-}\mathsf{Rel}(TX,TY)$$

by (1.2.iii): $\widetilde{\hat{T}r} := \iota\cdot\xi\cdot T\overrightarrow{r}\cdot\mathrm{can}^\circ_{X,Y}$ for all $r : X \to\!\!\!\!\!\!\shortmid\;\; Y$; elementwise, this formula translates to

$$(\hat{T}r)(\mathfrak{x},\mathfrak{y}) := \bigvee\{\xi\cdot T\overrightarrow{r}(\mathfrak{w})\,|\,\mathfrak{w}\in T(X\times Y), T\pi_1(\mathfrak{w})=\mathfrak{x}, T\pi_2(\mathfrak{w})=\mathfrak{y}\} \tag{1.4.i}$$

We write

$$\Phi(\xi) := \hat{T} = (\hat{T}_{X,Y}).$$

Calling the map ξ to be *monotone* if all maps

$$\mathsf{Set}(X,V) \to \mathsf{Set}(TX,V),\ \varphi \mapsto \xi \cdot T\varphi$$

are monotone, with $\mathsf{Set}(X,V)$ carrying the pointwise order of V (see 1.3), we have the following easy lemma:

1.4.1. **Lemma.** *Every map $\hat{T}_{X,Y}$ is monotone if and only if ξ is monotone.*

PROOF. Assuming first ξ to be monotone, we see that $r \leq s$ in $\mathsf{V}\text{-}\mathsf{Rel}(X,Y)$ implies $\overrightarrow{r} \leq \overrightarrow{s}$ and then $\xi \cdot T\overrightarrow{r} \leq \xi \cdot T\overrightarrow{s}$, which gives

$$\widetilde{\hat{T}r} = \iota \cdot \xi \cdot T\overrightarrow{r} \cdot \mathrm{can}^\circ_{X,Y} \leq \iota \cdot \xi \cdot T\overrightarrow{s} \cdot \mathrm{can}^\circ_{X,Y} = \widetilde{\hat{T}s}.$$

Conversely, $\varphi \leq \psi$ in $\mathsf{Set}(X,V)$ means $\iota \cdot \varphi \leq \iota \cdot \psi$ in $\mathsf{V}\text{-}\mathsf{Rel}(X,Y)$, hence $\hat{T}(\iota \cdot \varphi) \leq \hat{T}(\iota \cdot \psi)$ by hypothesis. Since $\overrightarrow{\iota \cdot \varphi} = \varphi$, this means

$$\iota \cdot \xi \cdot T\varphi \cdot \mathrm{can}^\circ_{X,1} = \widetilde{\hat{T}(\iota \cdot \varphi)} \leq \widetilde{\hat{T}(\iota \cdot \psi)} = \iota \cdot \xi \cdot T\psi \cdot \mathrm{can}^\circ_{X,1}.$$

But $\mathrm{can}^\circ_{X,1} \cdot \mathrm{can}_{X,1} = 1_{TX}$, so that $\xi \cdot T\varphi \leq \xi \cdot T\psi$ in $\mathsf{Set}(TX,V)$ follows. $\square$

1.4.2. **Definition.** (see [7, 3]) Recall that a commutative diagram

(1.4.ii)
$$\begin{array}{ccc} W & \xrightarrow{h_2} & Y \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

of Set maps is a *Beck-Chevalley square* (or *BC-square*) if $h_2 \cdot h_1^\circ = g^\circ \cdot f$ (or, equivalently, $h_1 \cdot h_2^\circ = f^\circ \cdot g$) in $\mathsf{V}\text{-}\mathsf{Rel}$ (or, equivalently, in $\mathbf{2}\text{-}\mathsf{Rel}$, with $\mathbf{2} = \{\bot < \top\}$). This means, equivalently, that (1.4.ii) is a weak pullback diagram in Set, so that the canonical map

$$W \longrightarrow X \times_Z Y$$

is surjective.

A Set-functor T *satisfies the Beck-Chevalley condition* (or *BC*) if it sends BC-squares to BC-squares:

$$h_2 \cdot h_1^\circ = g^\circ \cdot f \implies Th_2 \cdot (Th_1)^\circ = (Tg)^\circ \cdot Tf,$$

for all commutative diagrams (1.4.ii).

A natural transformation $\alpha : S \to T$ of Set-functors *satisfies BC* if all its naturality squares are BC-squares.

We can now prove:

1.4.3. **Proposition.** *If T satisfies the Beck-Chevalley condition, then, for any map $\xi : TV \to V$, the family $\hat{T} = \Phi(\xi)$ is right-whiskering*[4]*, that is:*

$$\hat{T}(r \cdot f) = \hat{T}r \cdot Tf$$

for all maps $f : Z \to X$ and V-relations $r : X \to\!\!\!\!\!\!\mapsto Y$.

Proof. Since the bottom square of (1.4.iii) is a pullback diagram, the top square is BC since the outer diagram is BC.

(1.4.iii)
$$\begin{tikzcd}
T(Z\times Y) \arrow[r, "T(f\times 1_Y)"] \arrow[d, "\mathrm{can}_{Z,Y}"'] \arrow[dd, bend right=60, "T\pi_Z"'] & T(X\times Y) \arrow[d, "\mathrm{can}_{X,Y}"] \arrow[dd, bend left=60, "T\pi_X"] \\
TZ\times TY \arrow[r, "Tf\times 1_{TY}"] \arrow[d] & TX\times TY \arrow[d] \\
TZ \arrow[r, "Tf"] & TX
\end{tikzcd}$$

Applying 1.1.1(7) twice we now obtain:

$$\begin{aligned}
\widetilde{\hat{T}(r\cdot f)} &= \iota\cdot\xi\cdot T(\overrightarrow{r\cdot f})\cdot \mathrm{can}^{\circ}_{Z,Y} \\
&= \iota\cdot\xi\cdot T\overrightarrow{r}\cdot T(f\times 1_Y)\cdot \mathrm{can}^{\circ}_{Z,Y} \\
&= \iota\cdot\xi\cdot T\overrightarrow{r}\cdot \mathrm{can}^{\circ}_{X,Y}\cdot (Tf\times 1_{TY}) \qquad \text{(BC)} \\
&= \widetilde{\hat{T}r}\cdot (Tf\times 1_{TY}) \\
&= \widetilde{\hat{T}r\cdot Tf}.
\end{aligned}$$

□

1.4.4. **Theorem.** *Let $T : \mathsf{Set} \to \mathsf{Set}$ be a functor satisfying BC and $\xi : TV \to V$ be a map. Then there is a uniquely determined algebraic lax extension $\hat{T}$ of T to V-Rel with structure map ξ if, and only if, ξ is monotone and satisfies the inequalities* (1.3.i).

Proof. The necessity of the conditions follows from 1.2.3, 1.3.1 and 1.4.1 (and does not require BC). Conversely, given a map ξ, any algebraic lax extension $\hat{T}$ of T with structure map ξ must necessarily satisfy (1.2.ii), by 1.2.3. Defining now $\hat{T} = \Phi(\xi)$ via (1.2.iii), $\hat{T}$ is monotone if ξ is monotone (1.4.1). Hence, the proof is complete once we have shown that the implications

(1) $k\cdot !_{T1} \le \xi\cdot Tk \;\Rightarrow\; \forall f\ (Tf \le \hat{T}f$ and $(Tf)^{\circ} \le \hat{T}(f^{\circ}))$,

[4]We use this term here differently from [7].

(2) $\otimes\cdot(\xi\times\xi)\cdot\mathrm{can}_{V,V}\leq\xi\cdot T\otimes\;\Rightarrow\forall r:X\,{\to\!\!\!\!\!\!\;\mapsto}\,Y,\;s:Y\,{\to\!\!\!\!\!\!\;\mapsto}\,Z\;(\hat{T}s\cdot\hat{T}r\leq\hat{T}(s\cdot r))$

hold whenever ξ is monotone and T satisfies BC.

(1) The diagram

$$\begin{tikzcd} X \arrow[r, "\langle 1_X, f\rangle"] \arrow[d, "!_X"'] & X\times Y \arrow[d, "\overrightarrow{f}"] \\ 1 \arrow[r, "k"] & V \end{tikzcd}$$

commutes for every map f. Furthermore, since $\mathrm{can}_{X,Y}\cdot T\langle 1_X,f\rangle=\langle 1_{TX},Tf\rangle$ one has $T\langle 1_X,f\rangle\leq\mathrm{can}^\circ_{X,Y}\cdot\langle 1_{TX},Tf\rangle$. Consequently, for all $\mathfrak{x}\in TX$, with $\mathfrak{y}:=Tf(\mathfrak{x})$ and $k\cdot !_{T1}\leq\xi\cdot Tk$ by hypothesis we obtain:

$$\begin{aligned}\widetilde{\hat{T}f}(\mathfrak{x},\mathfrak{y}) &= \iota\cdot\xi\cdot T\overrightarrow{f}\cdot\mathrm{can}^\circ_{X,Y}\cdot\langle 1_{TX},Tf\rangle(\mathfrak{x})\\ &\geq \iota\cdot\xi\cdot T\overrightarrow{f}\cdot T\langle 1_X,f\rangle(\mathfrak{x})\\ &= \iota\cdot\xi\cdot Tk\cdot T\,!_X(\mathfrak{x})\\ &\geq \iota\cdot k\cdot !_{T1}\cdot T\,!_X(\mathfrak{x})=k,\end{aligned}$$

which gives $\hat{T}f\geq Tf$. In addition, the proof of 1.2.3(3) shows

$$\hat{T}(f^\circ)=(\hat{T}f)^\circ\geq(Tf)^\circ.$$

(2) From the commutativity of the diagram

$$\begin{tikzcd} & T(X\times Z) \arrow[r, "\mathrm{can}_{X,Z}"] & TX\times TZ \\ & T(X\times Y\times Z) \arrow[u, "T(1_X\times !_Y\times 1_Z)"] \arrow[r, "\mathrm{can}_{X,Y,Z}"] \arrow[ur, phantom, "(a)"] \arrow[d, "T(1_X\times\delta_Y\times 1_Z)"'] & TX\times TY\times TZ \arrow[u, "1_{TX}\times !_{TY}\times 1_{TZ}"'] \arrow[d, "1_{TX}\times\delta_{TY}\times 1_{TZ}"] \\ T(V\times V) \arrow[d, "\mathrm{can}_{V,V}"'] & T(X\times Y\times Y\times Z) \arrow[l, "T(\overrightarrow{r}\times\overrightarrow{s})"'] \arrow[r, "\mathrm{can}_{X,Y,Y,Z}"] \arrow[ur, phantom, "(b1)"] \arrow[d, "\mathrm{can}_{X\times Y,Y\times Z}"'] & TX\times TY\times TY\times TZ \arrow[d, "1"] \\ TV\times TV & T(X\times Y)\times T(Y\times Z) \arrow[l, "T\overrightarrow{r}\times T\overrightarrow{s}"'] \arrow[r, "\mathrm{can}_{X,Y}\times\mathrm{can}_{Y,Z}"] \arrow[ur, phantom, "(b2)"] & TX\times TY\times TY\times TZ \end{tikzcd}$$

one obtains immediately:

(a) $T(1_X\times !_Y\times 1_Z)^\circ\cdot\mathrm{can}^\circ_{X,Z}=\mathrm{can}^\circ_{X,Y,Z}\cdot(1_{TX}\times !_{TY}\times 1_{TZ})^\circ$.

In addition, the combined square (b1)&(b2) satisfies BC. Indeed, for $(\mathfrak{u},\mathfrak{v})\in T(X\times Y)\times T(Y\times Z)$ and $(\mathfrak{x},\mathfrak{y},\mathfrak{z})\in TX\times TY\times TZ$ with $\mathrm{can}_{X,Y}(\mathfrak{u})=(\mathfrak{x},\mathfrak{y})$,

$\mathrm{can}_{Y,Z}(\mathfrak{v}) = (\mathfrak{y},\mathfrak{z})$, since T transforms the pullback diagram

$$\begin{tikzcd} X\times Y\times Z \arrow[r, "\pi_{2,3}"] \arrow[d, "\pi_{1,2}"'] & Y\times Z \arrow[d] \\ X\times Y \arrow[r] & Y \end{tikzcd}$$

into a BC square, we find $\mathfrak{w}\in T(X\times Y\times Z)$ with $T\pi_{1,2}(\mathfrak{w}) = (\mathfrak{x},\mathfrak{y})$ and $T\pi_{2,3}(\mathfrak{w}) = (\mathfrak{y},\mathfrak{z})$, which implies $\mathrm{can}_{X\times Y,Y\times Z}(T(1_X\times\delta_Y\times 1_Z)(\mathfrak{w})) = (\mathfrak{u},\mathfrak{v})$, $\mathrm{can}_{X,Y,Z}(\mathfrak{w}) = (\mathfrak{x},\mathfrak{y},\mathfrak{z})$. Hence:

(b) $\mathrm{can}_{X\times Y,Y\times Z}\cdot T(1_X\times\delta_Y\times 1_Z)\cdot\mathrm{can}^\circ_{X,Y,Z} \geq\leq (\mathrm{can}_{X,Y}\times\mathrm{can}_{Y,Z})^\circ\cdot(1_{TX}\times\delta_{TY}\times 1_{TZ})$,

with "$\leq$" holding by adjunction. Furthermore, from 1.1.1 one obtains by adjunction first

$$\iota\cdot(\overrightarrow{s\cdot r})\cdot(1_X\times !_Y\times 1_Z)\geq\iota\cdot\otimes\cdot(\overrightarrow{r}\times\overrightarrow{s})\cdot(1_X\times\delta_Y\times 1_Z)$$

and, with the monotonicity of ξ,

(c) $\iota\cdot\xi\cdot T(\overrightarrow{s\cdot r})\geq\iota\cdot\xi\cdot T\otimes\cdot T(\overrightarrow{r}\times\overrightarrow{s})\cdot T(1_X\times\delta_Y\times 1_Z)\cdot T(1_X\times !_Y\times 1_Z)^\circ$.

Combining (a)-(c) with the hypothesis we can complete the proof of the lax functoriality of $\hat{T}$:

$$\begin{array}{rcll}
\widetilde{\hat{T}(s\cdot r)} & = & \iota\cdot\xi\cdot T(\overrightarrow{s\cdot r})\cdot\mathrm{can}^\circ_{X,Z} & \\
 & \geq & \iota\cdot\xi\cdot T\otimes\cdot T(\overrightarrow{r}\times\overrightarrow{s})\cdot T(1_X\times\delta_Y\times 1_Z)\cdot & \\
 & & T(1_X\times !_Y\times 1_Z)^\circ\cdot\mathrm{can}^\circ_{X,Z} & \text{(c)} \\
 & \geq & \iota\cdot\otimes\cdot(\xi\times\xi)\cdot\mathrm{can}_{V,V}\cdot T(\overrightarrow{r}\times\overrightarrow{s})\cdot & \\
 & & T(1_X\times\delta_Y\times 1_Z)\cdot T(1_X\times !_Y\times 1_Z)^\circ\cdot\mathrm{can}^\circ_{X,Z} & \\
 & = & \iota\cdot\otimes\cdot(\xi\times\xi)\cdot(T\overrightarrow{r}\times T\overrightarrow{s})\cdot\mathrm{can}_{X\times Y,Y\times Z}\cdot & \\
 & & T(1_X\times\delta_Y\times 1_Z)\cdot\mathrm{can}^\circ_{X,Y,Z}\cdot(1_{TX}\times !_{TY}\times 1_{TZ})^\circ & \text{(a)} \\
 & \geq & (\iota\otimes\iota)\cdot(\xi\times\xi)\cdot(T\overrightarrow{r}\times T\overrightarrow{s})\cdot(\mathrm{can}^\circ_{X,Y}\otimes\mathrm{can}^\circ_{Y,Z})\cdot & \\
 & & (1_{TX}\times\delta_{TY}\times 1_{TZ})\cdot(1_{TX}\times !_{TY}\times 1_{TZ})^\circ & \text{(b)} \\
 & = & ((\iota\cdot\xi\cdot T\overrightarrow{r}\cdot\mathrm{can}^\circ_{X,Y})\otimes(\iota\cdot\xi\cdot T\overrightarrow{s}\cdot\mathrm{can}^\circ_{Y,Z}))\cdot & \\
 & & (1_{TX}\times\delta_{TY}\times 1_{TZ})\cdot(1_{TX}\times !_{TY}\times 1_{TZ})^\circ & \\
 & = & (\widetilde{\hat{T}r}\otimes\widetilde{\hat{T}s})\cdot(1_{TX}\times\delta_{TY}\times 1_{TZ})\cdot(1_{TX}\times !_{TY}\times 1_{TZ})^\circ & \\
 & = & \widetilde{\hat{T}s\cdot\hat{T}r}. &
\end{array}$$

$\square$

1.5 Left-whiskering algebraic lax extensions. Our next goal is to find conditions on the map $\xi: TV\to V$ that make $\hat{T} = \Phi(\xi)$ a lax extension not just of T, but of the monad $\mathbb{T} = (T,m,e)$, so that $e: 1\to\hat{T}$ and $m:\hat{T}\hat{T}\to\hat{T}$

become oplax natural transformations. It turns out that in order for us to verify these conditions, we actually need $\hat{T}$ not only to be right-whiskering (see 1.4.3) but also *left-whiskering*, so that

$$\hat{T}(h\cdot r)=Th\cdot\hat{T}r$$

for all $r: X{\to\!\!\!\!\!\!\mapsto} Y$, $h: Y\to Z$. Since $\hat{T}=\Phi(\xi)$ is dualizing (by the proof of 1.2.3(3)), the family $\hat{T}=\Phi(\xi)$ is left-whiskering precisely when it satisfies

$$\hat{T}(s\cdot j^\circ)=\hat{T}s\cdot(Tj)^\circ$$

for all $j: Y\to X$, $s: Y{\to\!\!\!\!\!\!\mapsto} Z$. Indeed, if $\hat{T}$ is left-whiskering, then

$$\hat{T}(s\cdot j^\circ)=\hat{T}((j\cdot s^\circ)^\circ)=(\hat{T}(j\cdot s^\circ))^\circ=(Tj\cdot\hat{T}s^\circ)^\circ=\hat{T}(s^{\circ\circ})\cdot(Tj)^\circ=\hat{T}s\cdot(Tj)^\circ;$$

likewise for the converse statement.

1.5.1. **Proposition.** *For any map $\xi: TV\to V$ with $\hat{T}=\Phi(\xi)$ right-whiskering, the following conditions are equivalent:*

(i) *$\hat{T}$ is monotone and left-whiskering;*

(ii) *for all maps $f: X\to Y$, $\varphi: X\to V$, $\psi: Y\to V$,*

$$\iota\cdot\psi\le\iota\cdot\varphi\cdot f^\circ\ \Rightarrow\ \iota\cdot\xi\cdot T\psi\le\iota\cdot\xi\cdot T\varphi\cdot(Tf)^\circ$$

(iii) *for all maps f,φ,ψ as in* (ii),

$$\forall y\in Y\ (\psi(y)\le\bigvee_{x\in f^{-1}y}\varphi(x))\ \Rightarrow\ \forall\mathfrak{y}\in TY\ (\xi(T\psi(\mathfrak{y}))\le\bigvee_{\mathfrak{x}\in(Tf)^{-1}\mathfrak{y}}\xi(T\varphi(\mathfrak{x}))).$$

Proof. (iii) is obviously just a pointwise rendering of (ii). For (i) $\Rightarrow$ (ii), if $\iota\cdot\psi\le\iota\cdot\varphi\cdot f^\circ$, the hypotheses guarantee $\hat{T}\iota\cdot T\psi\le\hat{T}\iota\cdot T\varphi\cdot(Tf)^\circ$. Since $!_{T1}\cdot\hat{T}\iota=\iota\cdot\xi$ (see the proof of 1.2.3), $\iota\cdot\xi\cdot T\psi\le\iota\cdot\xi\cdot T\varphi\cdot(Tf)^\circ$ follows. We are left with having to prove (ii) $\Rightarrow$ (i). By Lemma 1.4.1, condition (ii) certainly guarantees monotonicity of $\hat{T}$: just consider $f=1_X$. Now consider $j: Y\to X$ and $s: Y{\to\!\!\!\!\!\!\mapsto} Z$. Since $\iota\cdot\overrightarrow{s\cdot j^\circ}=\iota\cdot\overrightarrow{s}\cdot(j\times 1_Z)^\circ$ (see 1.1.1) implies $\iota\cdot\overrightarrow{s}\le\iota\cdot\overrightarrow{s\cdot j^\circ}\cdot(j\times 1_Z)$, with T monotone and right-whiskering one obtains on the one hand $\hat{T}\iota\cdot T\overrightarrow{s}\le\hat{T}\iota\cdot T(\overrightarrow{s\cdot j^\circ})\cdot T(j\times 1_Z)$ and then

$$\iota\cdot\xi\cdot T\overrightarrow{s}\cdot T(j\times 1_Z)^\circ\le\iota\cdot\xi\cdot T(\overrightarrow{s\cdot j^\circ}).$$

On the other hand, $\iota\cdot\overrightarrow{s\cdot j^\circ}=\iota\cdot\overrightarrow{s}\cdot(j\times 1_Z)^\circ$ implies with (ii)

$$\iota\cdot\xi\cdot T(\overrightarrow{s\cdot j^\circ})\le\iota\cdot\xi\cdot T\overrightarrow{s}\cdot T(j\times 1_Z)^\circ,$$

so that the two displayed inequalities are in fact equalities. Now

$$\begin{array}{rcl}\widetilde{\hat{T}(s\cdot j^\circ)} & = & \iota\cdot\xi\cdot T(\overrightarrow{s\cdot j^\circ})\cdot \mathrm{can}^\circ_{X,Z}\\ & = & \iota\cdot\xi\cdot T\overrightarrow{s}\cdot T(j\times 1_Z)^\circ\cdot\mathrm{can}^\circ_{X,Z}\\ & = & \iota\cdot\underbrace{\xi\cdot T\overrightarrow{s}}\cdot\mathrm{can}^\circ_{Y,Z}\cdot(Tj\times 1_{TZ})^\circ\\ & = & \hat{T}s\cdot(Tj)^\circ\end{array}$$

follows. $\square$

1.5.2. **Remark.** When $\hat{T}=\Phi(\xi)$ is monotone, the order dualizations of 1.5.1(ii) also holds:

$$\iota\cdot\varphi\cdot f^\circ\leq\iota\cdot\psi\ \Rightarrow\ \iota\cdot\xi\cdot T\varphi\cdot(Tf)^\circ\leq\iota\cdot\xi\cdot T\psi$$

for all f,φ,ψ. (Just write the premiss equivalently as $\iota\cdot\varphi\leq\iota\cdot\psi\cdot f$, and use the monotonicity of ξ and then again the adjunction.)

Here is another conceptual interpretation of the equivalent conditions of 1.5.1. By the Yoneda Lemma, there is a natural bijection

$$\{\alpha\,|\,\alpha:\mathsf{Set}(-,V)\to\mathsf{Set}(T-,V)\text{ nat. transf.}\}\to\mathsf{Set}(TV,V),\ \alpha\mapsto\alpha_V(1_V),$$

the inverse of which assigns to $\xi:TV\to V$ the natural transformation with components

$$\xi_X(\varphi)=\xi\cdot T\varphi\qquad(\varphi:X\to V);\tag{1.5.i}$$

by abuse of notation we write $\xi=(\xi_X)_{X\in\mathrm{ob}\mathsf{Set}}$. Calling a natural transformation α monotone if every component is monotone, then such transformations correspond to monotone maps ξ by definition:

$$\{\alpha\,|\,\alpha:\mathsf{Set}(-,V)\to\mathsf{Set}(T-,V)\text{ n. t., }\alpha\text{ monot.}\}\to\{\xi\in\mathsf{Set}(TV,V)\,|\,\xi\text{ monot.}\}.$$

Now, let us notice that, since every function $f:X\to Y$ has a right adjoint $f^\circ:Y\to X$ in $\mathsf{V}\text{-}\mathsf{Rel}$, the map

$$\begin{array}{ccc}\mathsf{V}\text{-}\mathsf{Rel}(Y,1) & \xrightarrow{\mathsf{V}\text{-}\mathsf{Rel}(f,1)} & \mathsf{V}\text{-}\mathsf{Rel}(X,1)\\ \cong & & \cong\\ \mathsf{Set}(Y,V) & \xrightarrow{\mathsf{Set}(f,V)} & \mathsf{Set}(X,V)\end{array}$$

has the left adjoint $\mathsf{V}\text{-}\mathsf{Rel}(f^\circ,1)$ in Ord; when written at the level of functions, that left adjoint is precisely $P_{\mathsf{V}}f$, with P_V denoting the V-*powerset functor*:

$$\mathsf{Set}(f,V)\vdash P_{\mathsf{V}}f:\mathsf{Set}(X,V)\to\mathsf{Set}(Y,V),$$

with $(P_{\mathsf{V}}f)(\varphi)(y)=(\iota\cdot\varphi\cdot f^\circ)(y)=\bigvee_{x\in f^{-1}y}\varphi(x)$ for all $\varphi:X\to V$, $y\in Y$.

So the question arises: which monotone natural transformations $\alpha:\mathsf{Set}(-,V)\to\mathsf{Set}(T-,V)$ produce simultaneously natural transformations

$\alpha : P_{\mathsf{V}} \to P_{\mathsf{V}}T$? Answer: precisely those that under the Yoneda bijection correspond to maps ξ satisfying the equivalent conditions (ii), (iii) of 1.5.1, as we show next.

1.5.3. **Proposition.** *A map $\xi : TV \to V$ satisfies the equivalent conditions* (ii), (iii) *of* 1.5.1 *if, and only if,* (1.5.i) *defines a natural transformation $P_{\mathsf{V}} \to P_{\mathsf{V}}T$ whose components are monotone.*

Proof. For ξ with 1.5.1(ii), we must show that

(1.5.ii)
$$\begin{tikzcd} P_{\mathsf{V}}X \arrow[r, "P_{\mathsf{V}}f"] \arrow[d, "\xi_X"'] & P_{\mathsf{V}}Y \arrow[d, "\xi_Y"] \\ P_{\mathsf{V}}TX \arrow[r, "P_{\mathsf{V}}Tf"'] & P_{\mathsf{V}}TY \end{tikzcd}$$

commutes. But with $\psi := (P_{\mathsf{V}}f)(\varphi)$, $\varphi \in P_{\mathsf{V}}X$, we have by 1.5.1(ii) in V-relational notation

$$\xi_Y((P_{\mathsf{V}}f)(\varphi)) = \iota \cdot \xi \cdot \psi \le \iota \cdot \xi \cdot T\varphi \cdot (Tf)^\circ = (P_{\mathsf{V}}Tf)(\xi_X(\varphi)),$$

and "$\ge$" holds by 1.5.2. Conversely, commutativity of (1.5.ii) in conjunction with monotonicity similarly implies condition (ii) of 1.5.1. $\square$

1.6 Algebraic lax monad extensions. For a monad $\mathbb{T} = (T, m, e)$, a commutative quantale $\mathsf{V} = (V, \otimes, k)$, and a map $\xi : TV \to V$ we first prove:

1.6.1. **Proposition.** *Let $\hat{T} = \Phi(\xi)$ be monotone and right- and left-whiskering. Then:*

1. *$e : 1_{\mathsf{V}\text{-}\mathsf{Rel}} \to \hat{T}$ is an oplax natural transformation if and only if $1_V \le \xi \cdot e_V$.*
2. *$m : \hat{T}\hat{T} \to \hat{T}$ is an oplax natural transformation if and only if $\xi \cdot T\xi \le \xi \cdot m_V$.*

(1.6.i)
$$\begin{tikzcd} V \arrow[r, "e_V"] \arrow[dr, "1_V"'] & TV \arrow[d, "\xi"] \arrow[dl, phantom, "\le"] \arrow[dr, phantom, "\le"] & TTV \arrow[l, "T\xi"'] \arrow[d, "m_V"] \\ & V & TV \arrow[l, "\xi"] \end{tikzcd}$$

Proof. (1) If $e : 1 \to \hat{T}$ is oplax, one obtains

$$\iota \cdot \xi \cdot e_V = !_{T1} \cdot \hat{T}\iota \cdot e_V \ge !_{T1} \cdot e_1 \cdot \iota = \iota \cdot 1_V,$$

hence $\xi\cdot e_V \geq 1_V$. Conversely, with $r : X \to\!\!\!\!\!\mapsto V$ this inequality implies

$$\begin{aligned}
\widetilde{e_Y\cdot r} &= \iota\cdot\overrightarrow{e_Y\cdot r}\\
&\leq \iota\cdot\xi\cdot e_V\cdot\overrightarrow{r}\cdot(1_X\times e_Y)^\circ\\
&= \iota\cdot\xi\cdot T\overrightarrow{r}\cdot e_{X\times Y}\cdot(1_X\times e_Y)^\circ\\
&\leq \iota\cdot\xi\cdot T\overrightarrow{r}\cdot \mathrm{can}^\circ_{X,Y}\cdot(e_X\times 1_{TY})\\
&= \widetilde{\hat{T}r}\cdot(e_X\times 1_{TY})\\
&= \widetilde{\hat{T}r\cdot e_X},
\end{aligned}$$

with the second inequality arising from $\mathrm{can}_{X,Y}\cdot e_{X\times Y} = (e_X\times 1_{TY})\cdot(1_X\times e_Y)$.

(2) Since $\hat{T}$ is right- and left-whiskering, for $m : \hat{T}\hat{T}\to\hat{T}$ oplax one obtains

$$\begin{aligned}
\iota\cdot\xi\cdot T\xi &= !_{T1}\cdot\hat{T}\iota\cdot T\xi = !\cdot\hat{T}(\iota\cdot\xi) = !\cdot\hat{T}(\,!\cdot\hat{T}\iota) = !\cdot T\,!\cdot\hat{T}\hat{T}\iota\\
&= !\cdot m_1\cdot\hat{T}\hat{T}\iota \leq !_{T1}\cdot\hat{T}\iota\cdot m_V = \iota\cdot\xi\cdot m_V,
\end{aligned}$$

hence $\xi\cdot T\xi\leq\xi\cdot m_V$. Conversely, let us first note that the commutative diagram

$$\begin{tikzcd}
TT(X\times Y) \arrow[r, "T\mathrm{can}_{X,Y}"] \arrow[d, "m_{X\times Y}"] & T(TX\times TY) \arrow[r, "\mathrm{can}_{TX,TY}"] & TTX\times TTY \arrow[r, "1\times m_Y"] & TTX\times TY \arrow[d, "m_X\times 1"]\\
T(X\times Y) \arrow[rrr, "\mathrm{can}_{X,Y}"] & & & TX\times TY
\end{tikzcd}$$

gives $m_{X\times Y}\cdot(T\mathrm{can}_{X,Y})^\circ\cdot\mathrm{can}^\circ_{TX,TY}\cdot(1_{TTX}\times m_Y)^\circ\leq\mathrm{can}^\circ_{X,Y}\cdot(m_X\times 1_{TY})$. Since $\hat{T}$ is right- and left-whiskering, from $\xi\cdot T\xi\leq\xi\cdot m_V$ one derives for every $r : X\to\!\!\!\!\!\mapsto Y$

$$\begin{aligned}
\widetilde{m_Y\cdot\hat{T}\hat{T}r} &= \widetilde{\hat{T}\hat{T}r}\cdot(1_{TTX}\cdot m_Y)^\circ\\
&= !\cdot\hat{T}\iota\cdot T(\overrightarrow{\hat{T}r})\cdot\mathrm{can}^\circ_{TX,TY}\cdot(1_{TTX}\times m_Y)^\circ\\
&= !\cdot\hat{T}(\iota\cdot\overrightarrow{\hat{T}r})\cdot\mathrm{can}^\circ_{TX,TY}(1_{TTX}\times m_Y)^\circ\\
&= !\cdot\hat{T}(\widetilde{\hat{T}r})\cdot\mathrm{can}^\circ_{TX,TY}\cdot(1_{TTX}\times m_Y)^\circ\\
&= !\cdot\hat{T}(\iota\cdot\xi\cdot T\overrightarrow{r}\cdot\mathrm{can}^\circ_{X,Y})\cdot\mathrm{can}_{TX,TY}\cdot(1_{TTX}\times m_Y)^\circ\\
&= \iota\cdot\xi\cdot T\xi\cdot TT\overrightarrow{r}\cdot(T\mathrm{can}_{X,Y})^\circ\cdot\mathrm{can}_{TX,TY}\cdot(1_{TTX}\times m_Y)^\circ\\
&\leq \iota\cdot\xi\cdot T\overrightarrow{r}\cdot m_{X\times Y}\cdot(T\mathrm{can}_{X,Y})^\circ\cdot\mathrm{can}_{TX,TY}\cdot(1_{TTX}\times m_Y)^\circ\\
&\leq \iota\cdot\xi\cdot T\overrightarrow{r}\cdot\mathrm{can}^\circ_{X,Y}\cdot(m_X\times 1_{TY})\\
&= \widetilde{\hat{T}r\cdot m_X}.
\end{aligned}$$

$\square$

Combining 1.4.4, 1.5.1 and 1.6.1 we may summarize our results, as follows.

1.6.2. **Theorem.** *Let $\mathbb{T} = (T, m, e)$ be a* Set*-monad satisfying BC,* $\mathsf{V} = (V, \otimes, k)$ *a quantale, and $\xi : TV \to V$ a map. Then*

$$(\hat{T}r)(\mathfrak{x}, \mathfrak{y}) = \bigvee\{\xi(T\overrightarrow{r}(\mathfrak{w})) \mid \mathfrak{w} \in T(X \times Y),\, T\pi_1(\mathfrak{w}) = \mathfrak{x},\, T\pi_2(\mathfrak{w}) = \mathfrak{y}\}$$

for all V*-relations $r : X \relbar\joinrel\mapsto Y$ and $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$ defines a left-whiskering lax extension of the monad $\mathbb{T} = (T, m, e)$ to* $\mathsf{V}\text{-}\mathsf{Rel}$ *if, and only if, the following conditions hold:*

(LIM) $k \cdot !_{T1} \le \xi \cdot Tk$ *and* $\otimes \cdot \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle \le \xi \cdot T\otimes$*;*
(LEM) $1_V \le \xi \cdot e_V$ *and* $\xi \cdot T\xi \le \xi \cdot m_V$*;*
(MON) $\xi_X : \mathsf{Set}(X, V) \to \mathsf{Set}(TX, V)$*,* $\varphi \mapsto \xi \cdot T\varphi$*, is monotone for all sets X;*
(NAT) $(\xi_X)_{X \in \mathrm{ob}\mathsf{Set}} : P_{\mathsf{V}} \to P_{\mathsf{V}}T$ *is a natural transformation.*

We refer to these four conditions as the *Lax Internal Monoid*, the *Lax Eilenberg-Moore*, the *Monotonicity* and the *Naturality* conditions on ξ, respectively. When the inequality signs in (LIM) or (LEM) may be replaced by equality signs, then we can drop "Lax" and write (IM) and (EM) respectively. Indeed, (EM) obviously just means that (V, ξ) is an object of the Eilenberg-Moore category $\mathsf{Set}^{\mathbb{T}}$, while (IM) then means that the monoid operations

$$k : (1, !_{T1}) \to (X, \xi),\ \otimes : (X, \xi) \times (X, \xi) \to (X, \xi)$$

live in the cartesian category $\mathsf{Set}^{\mathbb{T}}$ (=the monoidal category $(\mathsf{Set}^{\mathbb{T}}, \times, 1)$). But note that also in the absence of (EM), condition (IM) still makes $((V, \xi), \otimes, k)$ an internal monoid, namely of the larger category T-Alg of "T-algebras" $(X, \alpha : TX \to X)$ (that are not required to satisfy any further conditions) and "T-homomorphisms" (that are defined as in $\mathsf{Set}^{\mathbb{T}}$, which becomes a full subcategory of T-Alg.)

The impact of the more restrictive conditions (IM), (EM) (in the presence of (MON) and (NAT)) on the lax extension $\hat{T} = \Phi(\xi)$ will be discussed in the next section.

Here we just re-state Theorem 1.6.2 in terms of the bijective correspondence

$$\Phi : \{\xi \in \mathsf{Set}(TV, V) \mid (\mathrm{LIM})\,\&\,(\mathrm{MON})\} \to \mathsf{V}\text{-}\mathsf{AlgLxt}(T) \tag{1.6.ii}$$

given by Theorem 1.4.4 whose inverse assigns to an algebraic lax extension of the functor T to $\mathsf{V}\text{-}\mathsf{Rel}$ its structure map ξ. We note that Φ is actually an order isomorphism when we order lax extensions $\hat{T}$, $\hat{\hat{T}}$ of T by $\hat{T}r \le \hat{\hat{T}}r$ for all V-relations r (so that $1_T : (T, \hat{T}) \to (T, \hat{\hat{T}})$ becomes a morphism of lax extensions, see [7, III.3.4]).

1.6.3. **Corollary.** *The order-isomorphism* (1.6.ii) *restricts to a bijective correspondence between left-whiskering algebraic lax extensions to* $\mathsf{V}\text{-}\mathsf{Rel}$ *of the monad $\mathbb{T}$ and maps satisfying* (LIM)*,* (LEM)*,* (MON)*,* (NAT)*.*

1.6.4. **Remark.** For $(V, \otimes, k) = (V, \wedge, \top)$ a frame, in the presence of (MON) condition (LIM) automatically implies (IM). Indeed, $\xi \cdot Tk \leq k \cdot !_{T1}$ holds trivially when $k = \top$, and when one has also $\otimes = \wedge$, so that $\otimes \leq \pi_i$ $(i = 1, 2)$, monotonicity of ξ gives $\xi \cdot T\otimes \leq \xi \cdot T\pi_i$ and then $\xi \cdot T\otimes \leq \otimes \cdot \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle$.

2. Topological theories

2.1 Monoids in $\mathsf{Set}^{\mathbb{T}}$. For a monad $\mathbb{T} = (T, m, e)$, by 1.6.2 we are led to considering monoids in $\mathsf{Set}^{\mathbb{T}}$: for a monoid $\mathsf{V} = (V, \otimes, k)$ in Set one has a map $\xi : TV \to V$ satisfying (IM) and (EM) precisely when $\mathsf{V}_\xi = ((V, \xi), \otimes, k)$ is a monoid in the cartesian category $\mathsf{Set}^{\mathbb{T}}$. With the Set-monoid V we may associate the Set-monad $\mathbb{V} = (V \times (-), \tau, \kappa)$ with

$$\begin{array}{rcl} \kappa_X : X & \to & V \times X \\ x & \mapsto & (k, x) \end{array} \quad \text{and} \quad \begin{array}{rcl} \tau_X : V \times V \times X & \to & V \times X \\ (v, w, x) & \mapsto & (v \otimes w, x) \end{array} \tag{2.1.i}$$

whose Eilenberg-Moore category is precisely the category of (left) V-actions. Likewise, the $\mathsf{Set}^{\mathbb{T}}$-monoid V_ξ gives us the $\mathsf{Set}^{\mathbb{T}}$-monad $\mathbb{V}_\xi = ((V, \xi) \times (-), \tau, \kappa)$ where

$$(V, \xi) \times (X, \alpha) = (V \times X, T(V \times X) \xrightarrow{\mathrm{can}_{V,X}} TV \times TX \xrightarrow{\xi \times \alpha} V \times X),$$

whose Eilenberg-Moore category is the category of (left) V_ξ-actions in $\mathsf{Set}^{\mathbb{T}}$. Here we denoted the unit and multiplication of the monad $\mathbb{V}_\xi$ as for $\mathbb{V}$; in fact, it is easy to show that condition (IM) makes the maps κ_X and τ_X of (2.1.i) $\mathbb{T}$-homomorphisms. In other words, the monad $\mathbb{V}_\xi$ of $\mathsf{Set}^{\mathbb{T}}$ is a lifting of $\mathbb{V}$ along the forgetful $G^{\mathbb{T}} : \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}$ (see [7, II.3.8]).

2.1.1. **Proposition.** *For a monad $\mathbb{T}$ and a quantale V, maps $\xi : TV \to V$ satisfying* (IM) *and* (EM) *correspond bijectively to distributive laws of the monad $\mathbb{V}$ over $\mathbb{T}$. In particular, for any such ξ, the category of V_ξ-actions in $\mathsf{Set}^{\mathbb{T}}$ is monadic over* Set.

Proof. Since $(V, \otimes, k)$ may be recovered from the monad $\mathbb{V}$ as $\otimes = \tau_1$, $k = \kappa_1$, maps ξ with (IM), (EM) correspond bijectively to liftings of $\mathbb{V}$ along $G^{\mathbb{T}}$. But such liftings $\mathbb{V}_\xi$ correspond in turn bijectively to distributive laws $\chi : TV \to VT$ (where we have written V instead of $V \times (-)$), by [7, II.3.8.2]. Furthermore,

$$(\mathsf{Set}^{\mathbb{T}})^{\mathbb{V}_\xi} \to \mathsf{Set}$$

is just the forgetful functor of the Eilenberg-Moore category of the composite monad $\mathbb{V}\mathbb{T}$. □

2.1.2. **Remark.** The correspondence described by 2.1.1 associates with a distributive law $\chi : TV \to VT$ the map

$$\xi = (\ TV \xrightarrow{\chi_1} V \times T1 \xrightarrow{p_1} V\ ).$$

In fact, the free $(\mathbb{V}\mathbb{T})$-algebra over $(X, \alpha) \in \mathrm{ob}\mathsf{Set}^{\mathbb{T}}$ is generally given by

$$T(V \times X) \xrightarrow{\chi_X} V \times TX \xrightarrow{1_V \times \alpha} V \times X$$

which specializes to $p_1 \cdot \chi_1$, for $X = 1$.

2.1.3. **Examples.** (1) Let $\beta = (\beta, m, e)$ be the ultrafilter monad and $\mathsf{V} = \mathbf{2}$ the two-element chain. Since $\mathbf{2}$ is finite, $\beta 2 \cong 2$, with e_2 (and m_2) bijective and $\xi = e_2^{-1}$ the only map satisfying (EM); ξ also satisfies (IM). The Eilenberg-Moore category of the composite monad $\mathbf{2}\beta$ may be equivalently described as having objects compact Hausdorff spaces X that come equipped with a continuous projection $p : X \to X$ with $p \cdot p = p$, with morphisms continuous maps that commute with the attached projections.

(2) Let $\mathbb{L} = (L, m, e)$ be the word monad (i.e., the free-monoid monad) on Set, and let $\mathsf{V} = (V, \otimes, k)$ be any quantale. Then

$$\xi : LV \to V,\ (v_1, \ldots, v_n) \mapsto v_1 \otimes \cdots \otimes v_n,\ (\ ) \mapsto k,$$

satisfies (IM) and (EM). Since $\mathsf{Set}^{\mathbb{L}} \cong \mathsf{Mon}$, $(\mathbb{V}\mathbb{L})$-algebras may be described as monoids M which come with a V-action

$$a : V \times M \to M,\ (v, x) \mapsto vx,$$

which must be a homomorphism of monoids; hence, if we denote the binary operation of M by $*$, that means

$$(v \otimes w)(x * y) = (vx) * (wy)$$

for all $v, w \in V$, $x, y \in M$.

2.2 Algebraic lax extensions arising from monoids. By Theorem 1.6.2, maps $\xi : TV \to V$ satisfying (LIM), (LEM), (MON), (NAT) correspond to left-whiskering algebraic lax extensions $\hat{T}$ of the monad $\mathbb{T}$ to $\mathsf{V}\text{-}\mathsf{Rel}$. The question arises which additional properties $\hat{T}$ may enjoy if ξ satisfies even (IM), (EM).

2.2.1. **Theorem.** *Let $\xi : TV \to V$ satisfy* (LIM), (LEM), (MON), (NAT), *and let $\hat{T} = \Phi(\xi)$ be the algebraic lax extension with structure map ξ, with T satisfying BC. Then:*

(1) *$\hat{T}$ is flat (so that $\hat{T}f = Tf$ for all maps f) if and only if $k \cdot !_{T1} = \xi \cdot Tk$.*
(2) *$\hat{T}$ is a functor if and only if* (IM) *holds.*

(3) *If* $e : 1_{\mathsf{V\text{-}Rel}} \to \hat{T}$ *is a natural transformation, then* $\xi \cdot e_V = 1_V$*, with the converse statement holding when* $e : 1_{\mathsf{Set}} \to T$ *satisfies BC.*
(4) *If* $m : \hat{T}\hat{T} \to \hat{T}$ *is a natural transformation, then* $\xi \cdot T\xi = \xi \cdot m_V$*, with the converse statement holding when* $m : TT \to T$ *satisfies BC.*
(5) *If* $e : 1 \to T$ *and* $m : TT \to T$ *satisfy BC, then* $(\hat{T}, m, e)$ *is a monad on* $\mathsf{V\text{-}Rel}$ *if, and only if,* ξ *satisfies* (IM) *and* (EM)*.*

Proof. (1) If $\hat{T}$ is flat, one has in particular $\hat{T}u = Tu$ for u the identity map of 1. The first calculation of the proof of 1.3.1 shows that $k\cdot !_{T1} = \xi \cdot Tk$ follows. Conversely, assuming this equation to hold, it suffices to show $\hat{T}1_X = 1_{TX}$ for all X since $\hat{T}$ is right-whiskering. But since $\iota \cdot \overrightarrow{1_X} = \iota \cdot k\cdot !_X \cdot \delta_X^\circ$, from (NAT) one obtains

$$\begin{aligned} \widetilde{\hat{T}1_X} = \iota \cdot \xi \cdot T\overrightarrow{1_X} \cdot \mathrm{can}^\circ_{X,X} &= \iota \cdot \xi \cdot Tk \cdot T\,!_X \cdot (T\delta_X)^\circ \cdot \mathrm{can}^\circ_{X,X} \\ &= \iota \cdot k\cdot !_{T1} \cdot T\,!_X \cdot \delta^\circ_{TX} \\ &= !_{TX} \cdot \delta^\circ_{TX} = \widetilde{1_{TX}}. \end{aligned}$$

(2) That (IM) is a necessary condition for $\hat{T}$ being a functor follows from (1) and a re-examination of the proof of 1.3.1. Indeed, it suffices to make sure that the inequality (1.3.ii) is actually an equality. But since the algebraic lax extension $\hat{T}$ is right- and left-whiskering, from $\widetilde{\iota \otimes 1_V} = \iota \cdot \pi_1 \cdot (1_V \times \delta_V)^\circ$ one obtains

$$\begin{aligned} \hat{T}\widetilde{(\iota \otimes 1_V)} &= !_{T1} \cdot \hat{T}(\widetilde{\iota \otimes 1_V}) \cdot \mathrm{can}^\circ_{V\times V,V} \\ &= !_{T1} \cdot \hat{T}\iota \cdot T\pi_1 \cdot (T(1_V \times \delta_V))^\circ \cdot \mathrm{can}^\circ_{V\times V,V} \\ &= \iota \cdot \xi \cdot T\pi_1 \cdot \langle 1_{T(V\times V)}, T\pi_2 \rangle, \end{aligned}$$

which forces equality in (1.3.ii).

Conversely, in the presence of (IM) we must show that the lax extension $\hat{T}$ satisfies $\hat{T}(s \cdot r) \le \hat{T}s \cdot \hat{T}r$ for all V-relations $r : X \to\!\!\!\!\!\!+\; Y$, $s : Y \to\!\!\!\!\!\!+\; Z$. For that it suffices to recognize that the inequality (c) used in the proof of 1.4.4(2) is actually an equality, by 1.5.1, 1.5.2.

(3) Revisiting the proof of 1.6.1(1) we see that naturality of $e : 1 \to \hat{T}$ implies $\xi \cdot e_V = 1_V$, with the converse statement holding true if $e_{X\times Y} \cdot (1_X \times e_Y)^\circ = \mathrm{can}^\circ_{X,Y} \cdot (e_X \times 1_{TY})$. But when e satisfies BC, so that the outer rectangle of

$$\begin{tikzcd} X \times Y \arrow[r, "1_X \times e_Y"] \arrow[d, "e_{X\times Y}"'] & X \times TY \arrow[r, "\pi_1"] \arrow[d, "e_X \times 1_{TY}"] & X \arrow[d, "e_X"] \\ T(X \times Y) \arrow[r, "\mathrm{can}_{X,Y}"] & TX \times TY \arrow[r, "\pi_1"] & TY \end{tikzcd}$$

is BC, then also the left rectangle is BC since the right one is a pullback.

(4) Turning back to the proof of 1.6.1(2) now we see that naturality of $m : \hat{T}\hat{T} \to \hat{T}$ implies $\xi \cdot T\xi = \xi \cdot m_V$, and that the necessity of this condition is guaranteed if

$$m_{X\times Y} \cdot (T\mathrm{can}_{X,Y})^\circ \cdot \mathrm{can}^\circ_{TX,TY} \cdot (1_{TTX} \times m_Y)^\circ = \mathrm{can}^\circ_{X,Y} \cdot (m_X \times 1_{TY})$$

holds. But if m satisfies BC one can argue as in (3), considering the diagram

$$\begin{tikzcd} TT(X\times Y) \arrow[r, "T\mathrm{can}_{X,Y}"] \arrow[d, "m_{X\times Y}"'] & T(TX\times TY) \arrow[r, "\mathrm{can}_{TX,TY}"] & TTX\times TTY \arrow[r, "1\times m_Y"] & TTX\times TY \arrow[r, "\pi_1"] \arrow[d, "m_X\times 1_{TY}"'] & TTX \arrow[d, "m_X"] \\ T(X\times Y) \arrow[rrr, "\mathrm{can}_{X,Y}"] & & & TX\times TY \arrow[r, "\pi_1"] & TX \end{tikzcd}$$

(5) follows from (2)-(4). □

2.2.2. **Remarks.** (1) The proof of 2.2.1 shows that, without the requirement that T satisfies BC, for any $\xi : TV \to V$ satisfying (LIM), (LEM), (MON), (NAT), $\hat{T} = \Phi(\xi)$ is op-lax, that is:

$$\hat{T}1_X \leq 1_{TX},\ \hat{T}(s\cdot r) \leq \hat{T}s \cdot \hat{T}r$$

for all $r : X \rightarrow\!\!\!\!\!\!\mapsto Y$, $s : Y \rightarrow\!\!\!\!\!\!\mapsto Z$ in V-Rel.

(2) While both the unit and the multiplication of the word monad (2.1.3(2)) satisfy BC, for the ultrafilter monad only the multiplication satisfies BC. In fact, any Set-monad $\mathbb{T} = (T, m, e)$ with $T1 \cong 1$ and e satisfying BC must be isomorphic to the identity monad: see [7, II.1.12.4], [3].

2.2.3. **Definition.** ([6]) A *topological theory* is given by a Set-monad $\mathbb{T} = (T, m, e)$, a quantale $\mathsf{V} = (V, \otimes, k)$ and a map $\xi : TV \to V$ such that conditions (LIM), (LEM), (MON), (NAT) of 1.6.2 hold; the theory is *strict* if also (IM), (EM) hold true.

2.2.4. **Examples.** ([6])

(1) For every quantale V, the identity monad on Set and $\xi = 1_{\mathsf{V}}$ define a strict topological theory (the lax algebras of which are V-categories). Less trivially, the word monad $\mathbb{L}$ and the function ξ as defined in 2.1.3 define a strict topological theory (the lax algebras of which are V-multicategories; see [5]).

(2) If the quantale $\mathsf{V} = (V, \otimes, k)$ is (constructively) completely distributive (see [10, 7]), then the ultrafilter monad β together with the function $\xi : \beta V \to V$ with

$$\xi(\mathfrak{x}) = \bigvee_{A\in\mathfrak{x}} \bigwedge_{v\in A} v = \bigwedge_{A\in\mathfrak{x}} \bigvee_{v\in A} v$$

(for every ultrafilter $\mathfrak{x}$ on V) define a (generally non-strict) topological theory (which, for $\mathsf{V} = \mathbf{2}$, has topological spaces as lax algebras, and for $\mathsf{V} = [0,\infty]^{\mathrm{op}}$ approach spaces: see [5, 7]).

2.2.5. **Example.** (D. Hofmann, private communication) For any monoid H, consider the free H-act-monad ($X \mapsto H \times X$). Then, for every quantale $\mathsf{V} = (V, \otimes, k)$, the second projection $H \times V \to V$ defines a strict topological theory. Here is an example showing that, in general, one may have various choices for such a map $H \times V \to V$. Indeed, for $H = ([0,1], \cdot)$ and $V = [0,\infty]^{\mathrm{op}}$, the multiplication map

$$[0,1] \times [0,\infty] \to [0,\infty]$$

defines a strict topological theory as well.

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